Combinatorics A Solutions

1. There are five dots arranged in a line from left to right. Each of the dots is colored from one of five colors so that no 3 consecutive dots are all the same color. How many ways are there to color the dots?

   Proposed by: Nathan Bergman

   Answer: $\boxed{2800}$

   We use complimentary counting. There are $5^5 = 3125$ total ways to color the 5 dots. We now consider cases based on the maximum number of dots that are the same color. If the maximum is less than 3, we have no cases to consider, because none of these cases will have 3 consecutive dots of the same color. If the maximum is 3, then there are 5 ways to pick the color that appears 3 times, 3 ways to place these three dots so that they are consecutive, and $4^2 = 16$ ways to color the last two dots, which gives $5 * 3 * 16 = 240$. If there are 4 dots that are the same color, there are 5 ways to pick this color, 4 ways to arrange the dots so that 3 are consecutive, and 4 ways to choose the color the last dot for a total of $5 * 4 * 4 = 80$. Lastly, if all 5 dots are the same color, there are 5 ways to pick their color. This gives a desired answer of $3125 - 240 - 80 - 5 = \boxed{2800}$.

2. In an election between A and B, during the counting of the votes, neither candidate was more than 2 votes ahead, and the vote ended in a tie, 6 votes to 6 votes. Two votes for the same candidate are indistinguishable. In how many orders could the votes have been counted? One possibility is AABBABBABABA.

   Proposed by: Michael Gintz

   Answer: $\boxed{486}$

   For some sequence $(X_1, X_2, \ldots, X_{12})$, consider the subdivision $(X_1), (X_2, X_3), (X_{10}, X_{11}), (X_{12})$. Note that after an odd number of votes, one candidate will be ahead by one point. Consider then the possibilities for one of the pairs, such as (X_2, X_3). If A is ahead, the pair can be (A, B), (B, A), or (B, B), and if B is ahead, the pair can be (A, B), (B, A), or (A, A). Thus note that regardless of the previous elements, there are 3 choices for each pair. Note also that there are 2 options for the first vote, and there is only one option for the last vote (in order to tie the score). Thus there are a total of $2 \cdot 3^5 = \boxed{486}$ possibilities.

3. Alex starts at the origin O of a hexagonal lattice. Every second, he moves to one of the six vertices adjacent to the vertex he is currently at. If he ends up at X after 2018 moves, then let p be the probability that the shortest walk from O to X (where a valid move is from a vertex to an adjacent vertex) has length 2018. Then p can be expressed as $\frac{a^m - b}{c^n}$, where a, b, and c are positive integers less than 10; a and c are not perfect squares; and m and n are positive integers less than 10000. Find $a + b + c + m + n$.

   Proposed by: Eric Neyman

   Answer: $\boxed{4044}$

   Let H_k be the set of points that are accessible from O in k moves, but not $k - 1$ moves. Clearly $\cup_{k=1} n H_n$ is a hexagon of side length n centered at O, so H_{2018} is the perimeter of such a hexagon of side length 2018. Say that Alex lands on some specific edge. Clearly he can only arrive there by choosing between two adjacent directions 2018 times, so there are 2^{2018} ways to arrive at one of these points. Thus there are $3 \cdot 2^2 019$ ways to arrive at this hexagon, after overcounting the corners, which can only be arrived at in one way, but are counted twice, as the ends of two different edges. Thus there are a total of $3 \cdot 2^{2019} - 6$ ways to arrive at the hexagon. Thus our answer is $\frac{2^{2018}-1}{6^{2017}}$, so our answer is $\boxed{4044}$.

4. If a and b are selected uniformly from $\{0, 1, ..., 511\}$ with replacement, the expected number of 1's in the binary representation of $a + b$ can be written in simplest form as $\frac{m}{n}$. Compute $m + n$.

   Proposed by: Eric Neyman

   Answer: $\boxed{6143}$

   Let $f(n)$ be the answer for a and b selected from $\{0, 1, ..., 2^n - 1\}$. Note that $f(1) = \frac{3}{4}$. Now, consider $f(n + 1)$. We can think of a and b as being selected from $\{0, 1, ..., 2^n - 1\}$ and then with probability $\frac{1}{2}$ adding 2^n to their sum and with probability $\frac{1}{4}$ adding 2^{n+1}. If nothing is added, the number of ones is just the number of ones in $a + b$. If 2^{n+1} is added, it is that number plus another one in the 2^{n+1}'s place. If 2^n is added, that's an extra 1 if and only if $a + b$ has a 0 in its 2^n's place, i.e. $a + b < 2^n$. This happens with probability $\frac{2^n(2^n+1)}{2 \cdot 2^n \cdot 2^n} = \frac{2^n+1}{2^{n+1}}$. Thus, we have

$$f(n + 1) = f(n) + \frac{1}{4} + \frac{1}{2} \cdot \frac{2^n + 1}{2^{n+1}} = f(n) + \frac{1}{2}\left(1 + \frac{1}{2^{n+1}}\right).$$

   From here it is easy to find the explicit formula

$$f(n) = \frac{n + 1}{2} - \frac{1}{2^{n+1}}$$

   so $f(9) = 5 - \frac{1}{1024} = \frac{5119}{1024}$, giving us an answer of $\boxed{6143}$.

5. How many ways are there to color the 8 regions of a three-set Venn Diagram with 3 colors such that each color is used at least once? Two colorings are considered the same if one can be reached from the other by rotation and reflection.

   Proposed by: Sam Mathers

   Answer: $\boxed{1248}$

   We can first use Plya's Enumeration Theorem to get the number of ways of coloring the Venn Diagram with three colors given no restrictions. For 3 sets, A, B, and C, we have six possible symmetries, which fall into three distinct classes given by

$$(A, B, C)(AB, BC, AC)(ABC)(\varnothing) \quad \text{x2}$$

$$(A, B)(C)(AB)(AC, BC)(ABC)(\varnothing) \quad \text{x3}$$

$$(A)(B)(C)(AB)(AC)(BC)(ABC)(\varnothing) \quad \text{x1}$$

   Thus, by Plya's Theorem, the number of arrangements is

$$\frac{1}{6}(2 \cdot 3^4 + 3 \cdot 3^6 + 3^8) = 1485.$$

   However, to account for our restriction, we must subtract away all the possible ways of coloring our Venn Diagram with just 1 or 2 of the colors. We, thus first subtract away the number of ways of 2 coloring our Venn Diagram which is

$$\frac{1}{6}(2 \cdot 2^4 + 3 \cdot 2^6 + 2^8) = 80.$$

   However, we can use 2 colors in 3 different ways so we must subtract $3 \cdot 80$. But, for each color, we subtracted the uniform coloring with that color twice so we must add 3 back. Thus, in total, we have

$$1485 - 3 \cdot 80 + 3 = \boxed{1248}$$

   ways to color the Venn Diagram with 3 colors such that every color occurs at least once.

6. Michael is trying to drive a bus from his home, $(0,0)$ to school, located at $(6,6)$. There are horizontal and vertical roads at every line $x = 0, 1, \ldots, 6$ and $y = 0, 1, \ldots, 6$. The city has placed 6 roadblocks on lattice point intersections (x, y) with $0 \le x, y \le 6$. Michael notes that the only path he can take that only goes up and to the right is directly up from $(0,0)$ to $(0,6)$, and then right to $(6,6)$. How many sets of 6 locations could the city have blocked?

Proposed by: Zhuo Qun Song

Answer: $\boxed{263}$

Let $n = 6$, so that there are n roadblocks and the grid is from $(0,0)$ to (n,n).

First assume that every vertical and every horizontal street (other than $x = 0$ and $y = n$) has exactly one roadblock. Then, if there are roadblocks (a,b) and (c,d) with $a < c$ and $b < d$, we may take a path from $(0,0)$ to $(0,d)$ to (a,d) to (a,n) to (n,n) as well. Thus, the roadblocks must be precisely at $(i, n-i)$ for $i = 1, 2, \ldots, n$.

Next assume that there exists one vertical street $x = a$ and one horizontal street $y = b$ without a roadblock, with $a \neq 0$ and $b \neq n$. Then, Michael may go from $(0,0)$ to $(0,b)$ to (a,b) to (a,n) to (n,n), which is again a contradiction.

Finally, assume we are in the case where there is exactly one roadblock per horizontal street (and this condition may or may not be true of vertical streets). Let the roadblocks be at $(f(i), i)$ for some $i = 0, 1, \ldots, n-1$. We note that the necessary and sufficient conditions are the following: $f(n-1) = 1$, and $f(i) \le f(i+1) + 1$ for all $i = 0, 1, \ldots, n-2$. We will prove that they are necessary first, then that they are sufficient, and finally count the number of possible functions f that satisfy these conditions.

We have that $f(n-1) = 1$ is a necessary condition because otherwise Michael could take a path from $(0,0)$ to $(0, n-1)$ to $(1, n-1)$ to $(1,n)$ to (n,n). Similarly, if we have that $f(i) \ge f(i+1) + 2$ for some i, let i be maximal such that this is satisfied. In particular, this means that we have the inequalities $f(n-j) \le j < n$ for $j \le n-i-1$. Michael could then take a path from $(0,0)$ to $(0,i)$ to $(f(i)+1, i)$ to $(f(i)+1, i+1)$ to $(n, i+1)$ to (n,n).

Now, we prove that these conditions are sufficient. Assume that these conditions are satisfied and that there is another path other than from $(0,0)$ to $(0,6)$ to $(6,6)$. In particular, this means that the first time the path crosses $y = n-1$ is at $x > 1$ (because $(1, n-1)$ is a roadblock because of the conditions). There must then be some minimal k such that the first time the path crosses $y = k$ is at (d,k) with $d > f(k)$. Then, the path must go through $(d, k-1)$ with $d < f(k-1)$, so we have $f(k-1) \ge f(k) + 2$, which contradicts the second condition.

Finally, we count the number of such functions f. Define

$$g(i) = 1 - (f(n-i) - (n-i)),$$

so that the conditions become $g(1) = 1$, $g(i) \le i$ for $i = 2, \ldots, n$ and $g(i+1) \ge g(i)$ for $i = 2, \ldots, n-1$. This is precisely the Catalan bijection if we also define $g(n+1) = n+1$, so the number of such functions g is equal to

$$\frac{1}{n+1}\binom{2n}{n}$$

.

In conclusion, this is also the number of possible sets of roadblocks when there exists exactly one roadblock per vertical street, but we have counted the cases where there is both exactly one roadblock per horizontal street and exactly one roadblock per horizontal street, so our answer is

$$\frac{2}{n+1}\binom{2n}{n} - 1.$$

For $n = 6$, we get our answer of $\boxed{263}$.

7. Frankie the Frog starts his morning at the origin in $\mathbb{R}^2$. He decides to go on a leisurely stroll, consisting of $3^1 + 3^{10} + 3^{11} + 3^{100} + 3^{101} + 3^{110} + 3^{111} + 3^{1000}$ moves, starting with the 1st move. On the nth move, he hops a distance of

$$\max\{k \in \mathbb{Z} : 3^k | n\} + 1,$$

then turns 90° degrees counterclockwise. What is the square of the distance from his final position to the origin?

Proposed by: Michael Gintz

Answer: $\boxed{496016}$

Let us define a *stroll* as a sequence of hops, denoted by their distances, in which Frankie begins by facing in the positive-x direction, and after each hop Frankie is to rotate 90° counterclockwise. Let W be the stroll defined in the problem. Now say $A = a_0, \ldots, a_{1000}$ is a set of strolls, each of size $|W|$, such that the nth hop in a_k is 1 if $3^k | n$ and 0 otherwise.

If Frankie goes on each of the strolls in A, he will end in the same location as if he had gone on the stroll W, since the sum of the values of the nth hop over the elements of A is by definition the value of the nth hop in W. Therefore it suffices to determine the net movement of each of the strolls in A.

If we take any element of A and keep track of the direction Frankie is facing on his nonzero hops, we will see that he cycles through the four directions, so it suffices to determine the remainder when the number of nonzero hops Frankie makes is divided by 4 for each element of A. Let us define

$$S = \{1, 10, 11, 100, 101, 110, 111, 1000\}.$$

Consider some a_k. Note first that for all k we have

$$3^k > \sum_{i<k, i \in S} 3^i,$$

and each $i \geq k$ in S has 3^i as a multiple of 3^k. We can then use the fact that the value of $3^i/3^k \pmod 4$ is dependent only on the parity of $i - k$ to determine the number of nonzero hops that Frankie makes in $a_k \pmod 4$:

$$f(k) := \sum_{i \geq k, i \in S} (2i - 2k + 1) \pmod 4.$$

Now we have 4 possible outcomes based on the value of k.

- Say k is even and greater than 0. Then every value in S that is at least k has a corresponding value in S which differs from the first by 1, with the exception of 1000. Call the smaller one x. Then $(2x-2k+1+2x+2-2k+1) \equiv 0 \pmod 4$, so $f(k) \equiv 2000-2k+1 \equiv 1 \pmod 4$. The first direction Frankie faces on a nonzero hop must be the positive-x direction, as $3^k \equiv 1 \pmod 4$, so all a_k with even values of k which are greater than 0 are strolls whose net movement is one unit in the positive-x direction.
- Say k is odd but not in S. By the same logic as above, we have $f(k) \equiv 2000 - 2k + 1 \equiv 3 \pmod 4$. The fourth direction Frankie faces on a nonzero hop must be the negative-y direction, as Frankie always faces this direction on moves which are multiples of 4, so all a_k with odd values of k which are not in S are strolls whose net movement is one unit in the positive-y direction.

- Say k is 0. The value of $f(0)$ becomes a sum of equally many ones and threes, and thus is equivalent to $0 \pmod 4$, so a_0 is a stroll with no net movement.
- Say k is an odd element of S. Then by the same logic as before, we have $f(k) \equiv 2000 - 2k + 2 \equiv 0 \pmod 4$, so all of these strolls have no net movement.

Since there are 500 even numbers at least 2 and at most 1000, and there are 496 positive odd numbers less than 1000 which are not in S, Frankie moves 500 units in the positive-x direction and 496 units in the positive-y direction, and our answer by the Pythagorean Theorem is $500^2 + 496^2 = \boxed{496016}$.

8. Let S_5 be the set of permutations of $\{1, 2, 3, 4, 5\}$, and let C be the convex hull of the set

$$\{(\sigma(1), \sigma(2), \ldots, \sigma(5)) \mid \sigma \in S_5\}.$$

Then C is a polyhedron. What is the total number of 2-dimensional faces of C?

Proposed by: Michael Gintz

Answer: $\boxed{150}$

For sake of notation, let

$$U = \{(\sigma(1), \sigma(2), \ldots, \sigma(5)) \mid \sigma \in S_5\}.$$

For a face (of arbitrary dimension at least one) F of the convex hull of any polyhedral set, we have that there is some linear form $\mathbf{a} = (a_1, a_2, a_3, a_4, a_5)$ of real numbers such that F is the set of points $\mathbf{x}$ that solve the problem

$$\text{minimize} \quad \mathbf{a}^t\mathbf{x},$$

constrained under $\mathbf{x} \in C$. That is, F is the convex hull of the solution to the same problem constrained under $\mathbf{x} \in U$.

Now, let $\mathbb{T}$ be the set of faces of U, so that there is an association map $f : \mathbb{R}^5 \to \mathbb{T}$ that maps linear forms $\mathbf{a}$ to the corresponding faces. We show that

$$\mathbf{a}, \mathbf{b} \in f^{-1}(F) \quad \text{for some } F \in \mathbb{T}$$

$$\iff a_i - a_j \quad \text{and} \quad b_i - b_j \quad \text{have the same sign for all } i, j,$$

where sign refers to whether the quantity is positive, zero, or negative. We also show that $5 - \dim(F)$ is the number of distinct values in the a_i for some $\mathbf{a} \in f^{-1}(F)$.

Given these two facts, we note that if F has dimension two, any associated $\mathbf{a}$ must have either three elements equal or two pairs of two equal elements. In the first case, we have 10 ways of choosing the three equal elements, and then 6 ways of ordering those three elements along with the other two. In the second case, we have 15 ways of choosing the two pairs of two equal elements, and 6 ways of ordering those elements along with the fifth one. Thus, the answer is 150.

We now prove these two facts.

In the first statement, assume first that $f(\mathbf{a}) = f(\mathbf{b})$ and also that there is some pair i, j such that $a_i - a_j$ and $b_i - b_j$ are not of the same sign. In particular, assume without loss of generality that $b_i \geq b_j$ and $a_i < a_j$. Then, we have that $f(\mathbf{a})$ cannot contain any point $\mathbf{x}$ in U such that $x_i < x_j$, for the point $\mathbf{x}'$ with the ith and j^{th} coordinates swapped satisfies $\mathbf{a}^t\mathbf{x} > \mathbf{a}^t\mathbf{x}'$. Since the problem always has a solution because U is finite, there is at least one point $\mathbf{x}$ in both U and $f(\mathbf{a}) = f(\mathbf{b})$, and we must have that $x_i \geq x_j$. Further, since $\mathbf{x}$ is in U, we have that

$x_i > x_j$. But then we take $\mathbf{x}'$, which is $\mathbf{x}$ with the ith and j^{th} coordinates swapped, and since $b_i \geq b_j$, we have that $\mathbf{b}^t\mathbf{x} \geq \mathbf{b}^t\mathbf{x}'$ and so $\mathbf{x}'$ is also in U, which is a contradiction.

Next, assume that $f(\mathbf{a}) \neq f(\mathbf{b})$ but $a_i - a_j$ and $b_i - b_j$ have the same sign for all i, j. Then, for $\lambda \in [0, 1]$, let

$$F(\lambda) = f(\lambda\mathbf{b} + (1 - \lambda)\mathbf{a}),$$

so that $F(0) = f(\mathbf{a})$. Let U be the set of all values λ for which $F(\lambda) \neq f(\mathbf{a})$, and let λ' be the largest number that is less than or equal to every element of U (the infimum of U). Then, we have that $F(0) \subset F(\lambda')$, so there is some element in U and $F(\lambda')$, say $\mathbf{x}$ not in $F(0)$. Then, if we assume the second fact, then $F(\lambda')$ has higher dimension than $F(0)$ so it has fewer distinct values, but

$$\mathbf{t}_\lambda := \lambda\mathbf{b} + (1 - \lambda)\mathbf{a}$$

satisfies that $a_i - a_j$ and $(t_\lambda)_i - (t_\lambda)_j$ have the same sign for all i, j, which is a contradiction.

Finally, we prove the second fact, that $5 - \dim(F)$ is the number of distinct values in the a_i for some $\mathbf{a} \in f^{-1}(F)$. For any such a with k distinct values, we write a sequence of vectors $\mathbf{a}^1, \ldots, \mathbf{a}^5$ such that $\mathbf{a}^k$ has k distinct values, and that for any $i = 1, \ldots, 4$ and $\lambda \in [0, 1)$, the expression

$$\lambda(\mathbf{a}^{i+1}) + (1 - \lambda)(\mathbf{a}^i)$$

has i distinct values. This can be done by squeezing the two smallest distinct values together, for instance. Then, we have the chain of inclusions

$$F(\mathbf{a^5}) \subset F(\mathbf{a^4}) \subset \cdots \subset F(\mathbf{a^1}),$$

where each of the inclusions is strict and increases the dimension. Since $F(\mathbf{a}^1)$ has dimension 4 (it is all of C, which has dimension 4 because it is on the plane with sum of coordinates 15), we must have that $F(\mathbf{a^k})$ has dimension $5 - k$, as desired.

If you believe that any of these answers is incorrect, or that a problem had multiple reasonable interpretations or was incorrectly stated, you may appeal at http://tinyurl.com/PUMaCappeal2018. All appeals must be in by 1 PM to be considered.