Algebra A

1. Let x and y be positive real numbers that satisfy $(\log x)^2 + (\log y)^2 = \log(x^2) + \log(y^2)$. Compute the maximum possible value of $(\log xy)^2$.

2. Let $f(x) = x^2 + 4x + 2$. Let r be the difference between the largest and smallest real solutions of the equation $f(f(f(f(x)))) = 0$. Then $r = a^{\frac{p}{q}}$ for some positive integers a, p, q so a is square-free and p, q are relatively prime positive integers. Compute $a + p + q$.

3. Let Q be a quadratic polynomial. If the sum of the roots of $Q^{100}(x)$ (where $Q^i(x)$ is defined by $Q^1(x) = Q(x), Q^i(x) = Q(Q^{i-1}(x))$ for integers $i \geq 2$) is 8 and the sum of the roots of Q is S, compute $|\log_2(S)|$.

4. Let $\mathbb{N}_0$ be the set of non-negative integers. There is a triple (f, a, b), where f is a function from $\mathbb{N}_0$ to $\mathbb{N}_0$ and $a, b \in \mathbb{N}_0$, that satisfies the following conditions:

   1) $f(1) = 2$

   2) $f(a) + f(b) \leq 2\sqrt{f(a)}$

   3) For all $n > 0$, we have $f(n) = f(n-1)f(b) + 2n - f(b)$

   Find the sum of all possible values of $f(b + 100)$.

5. Let $\omega = e^{\frac{2\pi i}{2017}}$ and $\zeta = e^{\frac{2\pi i}{2019}}$. Let $S = \{(a, b) \in \mathbb{Z} \mid 0 \leq a \leq 2016, 0 \leq b \leq 2018, (a, b) \neq (0, 0)\}$. Compute $\prod_{(a,b)\in S} (\omega^a - \zeta^b)$.

6. A *weak binary representation* of a nonnegative integer n is a representation $n = a_0 + 2 \cdot a_1 + 2^2 \cdot a_2 + \ldots$ such that $a_i \in \{0, 1, 2, 3, 4, 5\}$. Determine the number of such representations for 513.

7. A doubly-indexed sequence $a_{m,n}$, for m and n nonnegative integers, is defined as follows.

   (a) $a_{m,0} = 0$ for all $m > 0$ and $a_{0,0} = 1$.

   (b) $a_{m,1} = 0$ for all $m > 1$, and $a_{1,1} = 1, a_{0,1} = 0$.

   (c) $a_{0,n} = a_{0,n-1} + a_{0,n-2}$ for all $n \geq 2$

   (d) $a_{m,n} = a_{m,n-1} + a_{m,n-2} + a_{m-1,n-1} - a_{m-1,n-2}$ for all $m > 0, n \geq 2$.

   Then there exists a unique value of x so $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a_{m,n} x^m}{3^{n-m}} = 1$. Find $\lfloor 1000x^2 \rfloor$.

8. For real numbers a and b, define the sequence $\{x_{a,b}(n)\}$ as follows: $x_{a,b}(1) = a$, $x_{a,b}(2) = b$, and for $n > 1$, $x_{a,b}(n+1) = (x_{a,b}(n-1))^2 + (x_{a,b}(n))^2$. For real numbers c and d, define the sequence $\{y_{c,d}(n)\}$ as follows: $y_{c,d}(1) = c$, $y_{c,d}(2) = d$, and for $n > 1$, $y_{c,d}(n+1) = (y_{c,d}(n-1) + y_{c,d}(n))^2$. Call (a, b, c) a good triple if there exists d such that for all n sufficiently large, $y_{c,d}(n) = (x_{a,b}(n))^2$. For some (a, b) there are exactly three values of c that make (a, b, c) a good triple. Among these pairs (a, b), compute the maximum value of $\lfloor 100(a + b) \rfloor$.