P U M $\therefore$ C

Algebra B

1. Let a, b be positive integers such that $a+b=10$. Let $\frac{p}{q}$ be the difference between the maximum and minimum possible values of $\frac{1}{a}+\frac{1}{b}$, where p and q are relatively prime positive integers. Compute $p+q$.

2. If x is a real number so $3^x = 27x$, compute $\log_3(\frac{3^{3^x}}{x^{3^3}})$.

3. Let x and y be positive real numbers that satisfy $(\log x)^2 + (\log y)^2 = \log(x^2) + \log(y^2)$. Compute the maximum possible value of $(\log xy)^2$.

4. Let $f(x) = x^2 + 4x + 2$. Let r be the difference between the largest and smallest real solutions of the equation $f(f(f(f(x)))) = 0$. Then $r = a^{\frac{p}{q}}$ for some positive integers a, p, q so a is square-free and p, q are relatively prime positive integers. Compute $a + p + q$.

5. Let Q be a quadratic polynomial. If the sum of the roots of $Q^{100}(x)$ (where $Q^i(x)$ is defined by $Q^1(x) = Q(x), Q^i(x) = Q(Q^{i-1}(x))$ for integers $i \geq 2$) is 8 and the sum of the roots of Q is S, compute $|\log_2(S)|$.

6. Let $\mathbb{N}_0$ be the set of non-negative integers. There is a triple (f, a, b), where f is a function from $\mathbb{N}_0$ to $\mathbb{N}_0$ and $a, b \in \mathbb{N}_0$, that satisfies the following conditions:

   1) $f(1) = 2$

   2) $f(a) + f(b) \leq 2\sqrt{f(a)}$

   3) For all $n > 0$, we have $f(n) = f(n-1)f(b) + 2n - f(b)$

   Find the sum of all possible values of $f(b + 100)$.

7. Let $\omega = e^{\frac{2\pi i}{2017}}$ and $\zeta = e^{\frac{2\pi i}{2019}}$. Let $S = \{(a, b) \in \mathbb{Z} \mid 0 \leq a \leq 2016, 0 \leq b \leq 2018, (a, b) \neq (0, 0)\}$. Compute $\prod_{(a,b)\in S} (\omega^a - \zeta^b)$.

8. A *weak binary representation* of a nonnegative integer n is a representation $n = a_0 + 2 \cdot a_1 + 2^2 \cdot a_2 + \ldots$ such that $a_i \in \{0, 1, 2, 3, 4, 5\}$. Determine the number of such representations for 513.