Algebra B Solutions

1. Let a, b be positive integers such that $a+b=10$. Let $\frac{p}{q}$ be the difference between the maximum and minimum possible values of $\frac{1}{a}+\frac{1}{b}$, where p and q are relatively prime. Compute $p+q$.

   Answer: $\boxed{77}$

   Proposed by: Matthew Kendall

   Since $b = 10-a$, $\frac{1}{a}+\frac{1}{b} = \frac{1}{a}+\frac{1}{10-a} = \frac{10}{a(10-a)} = \frac{10}{-a^2+10a}$. For the maximum, we wish to minimize the value of the denominator. That is achieved at the axis of symmetry of the parabola, when $a=5$, making $\frac{1}{a}+\frac{1}{b}=\frac{2}{5}$. The minimum is achieved when the denominator is maximized, which is when a is as far from the axis of symmetry as possible: $a=1$ or $a=9$. Either value gives $\frac{1}{a}+\frac{1}{b}=\frac{1}{9}+\frac{1}{1}=\frac{10}{9}$.

   Hence, the difference of the minimum and maximum is $\frac{10}{9}-\frac{2}{5}=\frac{32}{45}$, making the answer $32+45=\boxed{77}$.

2. If x is a real number so $3^x = 27x$, compute $\log_3(\frac{3^{3^x}}{x^{3^3}})$.

   Answer: $\boxed{81}$

   Proposed by: Frank Lu

   We plug in the condition that we were given initially to get a value of $\log_3(\frac{3^{27x}}{x^{27}})$. We can simplify this by using the equality again to get $\log_3(\frac{(3^x)^{27}}{x^{27}}) = \log_3(\frac{27^{27} * x^{27}}{x^{27}}) = \log_3(27^{27}) = 27 * 3 = \boxed{81}$.

3. Let x and y be positive real numbers that satisfy $(\log x)^2 + (\log y)^2 = \log(x^2) + \log(y^2)$. Compute the maximum possible value of $(\log xy)^2$.

   Answer: $\boxed{16}$

   Proposed by: Matthew Kendall

   Let $u = \log x$ and $v = \log y$. Then $u^2+v^2 = 2u+2v$. Completing the square gives $(u-1)^2 + (v-1)^2 = 2$, so the equation given is a circle of radius $\sqrt{2}$ centered at $(1,1)$ on the uv plane.

   Let $\log xy = u+v = k$, so we wish to maximize k^2. Note that the line $u+v$ is tangent to the circle when the origin is a distance of 0 or $2\sqrt{2}$ from the line. The latter gives $u=v=2$, so $k=4$, making the maximum $k^2 = \boxed{16}$.

4. Let $f(x) = x^2+4x+2$. Let r be the difference between the largest and smallest real solutions of the equation $f(f(f(f(x)))) = 0$. Then $r = a^{\frac{p}{q}}$ for some positive integers a, p, q so a is square-free and p, q are relatively prime positive integers. Compute $a+p+q$.

   Answer: $\boxed{35}$

   Proposed by: Kevin Feng

   Note that $f(x) = x^2+4x+2 = (x+2)^2-2$. Then $f^2(x) = ((x+2)^2-2)+2)^2-2 = (x+2)^4-2$. It is easy to see by induction that $f^n(x) = (x+2)^{2^n}-2$, so $f^4(x) = (x+2)^{2^4}-2$.

   Then the real solutions to $f^4(x) = 0$ are at $x+2 = \pm\sqrt[16]{2}$, or $x = -2\pm\sqrt[16]{2}$. Hence, the difference between the two of them are $2\sqrt[16]{2} = 2^{\frac{17}{16}}$, which gives us an answer of $2+16+17 = \boxed{35}$.

5. Let Q be a quadratic polynomial. If the sum of the roots of $Q^{100}(x)$ (where $Q^i(x)$ is defined by $Q^1(x) = Q(x), Q^i(x) = Q(Q^{i-1}(x))$ for integers $i \geq 2$) is 8 and the sum of the roots of Q is S, compute $|\log_2(S)|$.

Answer: $\boxed{96}$

Proposed by: Matthew Kendall

Let the sum of the roots of $Q^j(x)$ be S_j for $j = 1, \ldots, 2019$. Our claim is $S_{j+1} = 2S_j$. Let $Q(x) = a(x-r)(x-s)$, where r and s are the roots of Q. Note that

$$Q^{j+1}(x) = a(Q^j(x) - r)(Q^j(x) - s),$$

so the solutions to $Q^{j+1}(x) = 0$ are the solutions to $Q^j(x) = r$ and $Q^j(x) = s$. Since the degree of Q^j is at least 1, the sum of the roots to $Q^j(x) = r$ and $Q^j(x) = s$ are both S_j, so $S_{j+1} = S_j + S_j = 2S_j$.

From our recursion we get $S_{100} = 2^{99}S_1$. Therefore, $S_1 = \frac{8}{2^{99}}$ and $|\log_2(S)| = \boxed{96}$.

6. Let $\mathbb{N}_0$ be the set of non-negative integers. There is a triple (f, a, b), where f is a function from $\mathbb{N}_0$ to $\mathbb{N}_0$ and $a, b \in \mathbb{N}_0$, that satisfies the following conditions:

   1) $f(1) = 2$

   2) $f(a) + f(b) \le 2\sqrt{f(a)}$

   3) For all $n > 0$, we have $f(n) = f(n-1)f(b) + 2n - f(b)$

   Find the sum of all possible values of $f(b + 100)$.

   Answer: $\boxed{10201}$

   Proposed by: Rahul Saha

   We'll focus on condition 2.

   By AM-GM (or squaring and rearranging),

$$2\sqrt{f(a)f(b)} \le f(a) + f(b) \le 2\sqrt{f(a)}$$

   which implies $f(b) \le 1$. Since $f(b)$ is as integer we must have $f(b) = 0, 1$.

   Substituting in condition (3) gives us the possibilities $f(n) = 2n$ for $n > 0$ (for $f(b) = 0$) and a recursion which easily amounts to $f(n) = n^2 + 1$.

   For the first function, since $f(n) = 2n$ for $n > 0$ and $f(b) = 0$, we must necessarily have $b = 0$. So $f(b + 100) = f(100) = 200$.

   In the second case, similarly $b = 0$ and $f(b + 100) = 100^2 + 1 = 10001$.

   Summing gives us the answer $\boxed{10201}$.

   Quick check: In both cases, if we have $a = b = 0$, condition 2 holds. Condition 1 works for both functions too. So our functions do satisfy the problem's statement.

7. Let $\omega = e^{\frac{2\pi i}{2017}}$ and $\zeta = e^{\frac{2\pi i}{2019}}$. Let $S = \{(a, b) \in \mathbb{Z} \mid 0 \le a \le 2016, 0 \le b \le 2018, (a, b) \neq (0, 0)\}$. Compute $\prod_{(a,b)\in S} (\omega^a - \zeta^b)$.

   Answer: $\boxed{4072323}$

   Proposed by: Frank Lu

   First, fix a. Note that $\prod_{b=0}^{2018} (x - \zeta^b) = x^{2019} - 1$. Hence, if $a \neq 0$, $\prod_{b=0}^{2018} (\omega^a - \zeta^b) = \omega^{2019a} - 1$. For $a = 0$, we have that this is $\prod_{b=1}^{2018} (1 - \zeta^b) = 2019$, since $\prod_{b=1}^{2018} (x - \zeta^b) = \prod_{b=0}^{2018} (x - \zeta^b)/(x-1) = \sum_{b=0}^{2018} x^b$.

   Thus, our product becomes $\prod_{a=1}^{2016} (\omega^{2019a} - 1) * 2019$. But note that this then becomes $2017 * 2019$, since the ω^{2019a} are just a permutation of the 2017th roots of unity besides 1 (as 2017 and 2019 are relatively prime), which is then just $\boxed{4072323}$.

8. A *weak binary representation* of a nonnegative integer n is a representation $n = a_0 + 2 \cdot a_1 + 2^2 \cdot a_2 + \ldots$ such that $a_i \in \{0, 1, 2, 3, 4, 5\}$. Determine the number of such representations for 513.

   Answer: $\boxed{3290}$

   Proposed by: Frank Lu

   Let $N(k)$ be the number of such representations for k. We know that $N(0) = 1, N(1) = 1, N(2) = 2, N(3) = 2$, and $N(4) = 4$. We can see, based on the choice of a_0, that $N(2k) = N(2k+1) = N(k) + N(k-1) + N(k-2)$. To make use of this recurrence relation, we define two sequences. First, define $x_k = N(2k)$. Observe then that $x_{2k} - x_{2k-1} = N(4k) - N(4k-2) = N(2k) - N(2k-3) = x_k - x_{k-2}$, and that $x_{2k+1} - x_{2k} = x_k - x_{k-1}$ by a similar token. Now, let $y_k = x_k - x_{k-1}$. Then, our recurrence relation becomes $y_{2k+1} = y_k$ and that $y_{2k} = y_k + y_{k-1}$. From our earlier cases before we see that $y_1 = 1$ and $y_2 = 2$. Based on the recurrence in the odd case, we see that $y_{2^k-1} = 1$ for each integer i.

   Claim: $\sum_{i=2^{k-1}-1}^{2^k-2} y_i = 3^{k-1}$.

   Proof: We inductively show this. For $k = 2$ we can easily verify this. Now, given the k case, note that $\sum_{i=2^k-1}^{2^{k+1}-2} y_i = \sum_{i=2^{k-1}-1}^{2^k-2} y_{2i+1} + \sum_{i=2^{k-1}}^{2^k-1} y_{2i}$. Using our recurrence, this becomes $\sum_{i=2^{k-1}-1}^{2^k-1} y_i + \sum_{i=2^{k-1}}^{2^k-1} y_i + \sum_{i=2^{k-1}}^{2^k-1} y_{i-1}$. But knowing that $y_{2^k-1} = y_{2^{k-1}-1} = 1$ yields that this is just $3 * \sum_{i=2^{k-1}-1}^{2^k-2} y_i = 3^{k-1} * 3 = 3^k$, proving the inductive case and proving the claim.

   Finally, observe that $N(513) = N(512) = x_{256} = x_0 + \sum_{i=1}^{256} y_i$, which using our claim is just $1 + 3 + 3^2 + \ldots + 3^7 + y_{255} + y_{256}$. A final observation that $y_{2*k} = k + 1$ yields the answer $(3^8 - 1)/2 + 1 + 9 = \boxed{3290}$.