P U M .·. C

Combinatorics A

1. Prinstan Trollner and Dukejukem are competing at the game show WASS. Both players spin a wheel which chooses an integer from 1 to 50 uniformly at random, and this number becomes their score. Dukejukem then flips a weighted coin that lands heads with probability 3/5. If he flips heads, he adds 1 to his score. A player wins the game if their score is higher than the other player's score. The probability Dukejukem defeats the Trollner to win WASS equals m/n where m, n are coprime positive integers. Compute $m + n$.

2. Keith has 10 coins labeled 1 through 10, where the ith coin has weight 2^i. The coins are all fair, so the probability of flipping heads on any of the coins is $\frac{1}{2}$. After flipping all of the coins, Keith takes all of the coins which land heads and measures their total weight, W. If the probability that $137 \leq W \leq 1061$ is m/n for coprime positive integers m, n, determine $m + n$.

3. Marko lives on the origin of the Cartesian plane. Every second, Marko moves 1 unit up with probability 2/9, 1 unit right with probability 2/9, 1 unit up and 1 unit right with probability 4/9, and he doesn't move with probability 1/9. After 2019 seconds, Marko ends up on the point (A, B). What is the expected value of $A \cdot B$?

4. Kelvin and Quinn are collecting trading cards; there are 6 distinct cards that could appear in a pack. Each pack contains exactly one card, and each card is equally likely. Kelvin buys packs until he has at least one copy of every card, then he stops buying packs. If Quinn is missing exactly one card, the probability that Kelvin has at least two copies of the card Quinn is missing is expressible as m/n for coprime positive integers m, n. Determine $m + n$.

5. A candy store has 100 pieces of candy to give away. When you get to the store, there are five people in front of you, numbered from 1 to 5. The ith person in line considers the set of positive integers congruent to i modulo 5 which are at most the number of pieces of candy remaining. If this set is empty, then they take no candy. Otherwise they pick an element of this set and take that many pieces of candy. For example, the first person in line will pick an integer from the set $\{1, 6, ..., 96\}$ and take that many pieces of candy. How many ways can the first five people take their share of candy so that after they are done there are at least 35 pieces of candy remaining?

6. The Nationwide Basketball Society (NBS) has 8001 teams, numbered 2000 through 10000. For each n, team n has $n + 1$ players, and in a sheer coincidence, this year each player attempted n shots and on team n, exactly one player made 0 shots, one player made 1 shot, ..., one player made n shots. A player's *field goal percentage* is defined as the percentage of shots that the player made, rounded to the nearest tenth of a percent. (For instance, 32.45% rounds to 32.5%.) A player in the NBS is randomly selected among those whose field goal percentage is 66.6%. If this player plays for team k, what is the probability that $k \geq 6000$?

7. In the country of PUMACsboro, there are n distinct cities labelled 1 through n. There is a rail line going from city i to city j if and only if $i < j$; you can only take this rail line from city i to city j. What is the smallest possible value of n, such that if each rail line's track is painted orange or black, you can always take the train between 2019 cities on tracks that are all the same color? (This means there are some cities $c_1, c_2, \ldots, c_{2019}$, such that there is a rail line going from city c_i to c_{i+1} for all $1 \leq i \leq 2018$, and their rail lines' tracks are either all orange or all black.)

8. Let S_n be the set of points $(x/2, y/2) \in \mathbb{R}^2$ such that x, y are odd integers and $|x| \leq y \leq 2n$. Let T_n be the number of graphs G with vertex set S_n satisfying the following conditions:

- G has no cycles.
- If two points share an edge, then the distance between them is 1.
- For any path $P = (a, \ldots, b)$ in G, the smallest y-coordinate among the points in P is either that of a or that of b. However, multiple points may share this y-coordinate.

Find the 100th-smallest positive integer n such that the units digit of T_{3n} is 4.