Geometry A Solutions

1. A right cone in xyz-space has its apex at $(0,0,0)$, and the endpoints of a diameter on its base are $(12,13,-9)$ and $(12,-5,15)$. The volume of the cone can be expressed as $a\pi$. What is a?

   Proposed by Nathan Bergman and Jacob Wachspress.

   Answer: $\boxed{975}$

   Solution: The center of the base is the midpoint of the diameter $(12,4,3)$. Then the height of the cone is 13 and the radius of the cone is 15, so the volume is 975π.

2. Let ΔABC be a triangle with circumcenter O and orthocenter H. Let D be a point on the circumcircle of ABC such that $AD \perp BC$. Suppose that $AB = 6$, $DB = 2$, and the ratio $\frac{\text{area}(\Delta ABC)}{\text{area}(\Delta HBC)} = 5$. Then, if OA is the length of the circumradius, then OA^2 can be written in the form $\frac{m}{n}$, where m, n are relatively prime nonnegative integers. Compute $m+n$.

   Note: The circumradius is the radius of the circumcircle.

   Proposed by Oliver Thakar.

   Answer: $\boxed{29}$

   Solution: The key observation here is that ΔBDC and ΔBHC are in fact congruent. Then, $\frac{AB\times AC}{HB\times HC} = \frac{AB\times AC}{DB\times DC} = 5$.

   Because $ABCD$ is an orthogonal cyclic quadrilateral, we get the following relations: $AB^2 + DC^2 = 4OA^2$ and $AC^2 + DB^2 = 4OA^2$ In combination with $AB \times AC = 5DB \times DC$, we can solve for OA. We know that $\frac{3}{5}AC = DC$. Thus, $AB^2 + DC^2 = AC^2 + DB^2$ reduces to $32 + \frac{9}{25}AC^2 = AC^2$, so that $AC^2 = 50$. That makes $4OA^2 = 50 + 4$, so $OA^2 = \frac{27}{2}$, or $OA = 3\sqrt{\frac{3}{2}}$.

3. Suppose we choose two real numbers $x, y \in [0,1]$ uniformly at random. Let p be the probability that the circle with center (x,y) and radius $|x-y|$ lies entirely within the unit square $[0,1]\times[0,1]$. Then p can be written in the form $\frac{m}{n}$, where m and n are relatively prime nonnegative integers. Compute m^2+n^2.

   Proposed by Sam Mathers.

   Answer: $\boxed{10}$

   Solution: The key observation here is that ΔBDC and ΔBHC are in fact congruent. Then, $\frac{AB\times AC}{HB\times HC} = \frac{AB\times AC}{DB\times DC} = 5$.
   First, suppose $x > y$, then we have the conditions $x - y < y$ and $x - y < 1 - x$. The point of intersection of these two inequalities is when $x = 2y$ and $y = 2x - 1$ so $x = \frac{2}{3}$ and $y = \frac{1}{3}$. Thus, the acceptable region for (x,y) is within the triangle with vertices $(0,0)$, $(\frac{2}{3}, \frac{1}{3})$, and $(1,1)$. This has area $\frac{1}{6}$. Multiplying this by two since we also have the case $y > x$, we get an area of $\frac{1}{3}$ so the answer is 10.

4. Let $BC = 6, BX = 3, CX = 5$, and let F be the midpoint of BC. Let $AX \perp BC$ and $AF = \sqrt{247}$. If AC is of the form $\sqrt{b}$ and AB is of the form $\sqrt{c}$ where b and c are nonnegative integers, find $2c + 3b$.

   Proposed by Alan Chung. Solution by Aleksa Milojević.

   Answer: $\boxed{1288}$

   Solution: Consider the circles C_1 with center B and radius $BX = 3$ and C_2 with center C and radius $CX = 5$. Their radical axis is a line through X perpendicular to BC, i.e. the line

AX. As A is on this radical axis, we get that its power w.r.t. circles C_1 and C_2 is the same: $AB^2 - BX^2 = AC^2 - CX^2$.

On the other hand, by the median formula in triangle ABC we have that $AF^2 = \frac{1}{2}BA^2 + \frac{1}{2}CA^2 - \frac{1}{4}BC^2$. The two equations we have got are enough to find AB^2 and AC^2. Namely:

$$AB^2 + AC^2 = 2(AF^2 + \frac{1}{4}BC^2) = 2 \cdot 256 = 512$$
$$AB^2 - AC^2 = BX^2 - CX^2 = -16$$

By solving this system we get $AB^2 = c = 248$ and $AC^2 = b = 264$. Therefore, $2c + 3b = 1288$.

5. Let Γ be a circle with center A, radius 1 and diameter BX. Let Ω be a circle with center C, radius 1 and diameter DY, where X and Y are on the same side of AC. Γ meets Ω at two points, one of which is Z. The lines tangent to Γ and Ω that pass through Z cut out a sector of the plane containing no part of either circle and with angle $60°$. If $\angle XYC = \angle CAB$ and $\angle XCD = 90°$, then the length of XY can be written in the form $\frac{\sqrt{a}+\sqrt{b}}{c}$ for integers a, b, c where $\gcd(a, b, c) = 1$. Find $a + b + c$.

   Proposed by Zachary Stier.

   Answer: $\boxed{16}$

   Solution: Let the circles have radii a, c and let the angle at Z be θ. We first compute AC. $\angle AZC = \theta + 2(90° - \theta) = 180° - \theta$ so the Law of Cosines gives $b = AC = \sqrt{a^2 + c^2 + 2ab\cos\theta}$.

   The given angle conditions make $AXYC$ cyclic, and the right angle makes XY the diameter (of length d) of the circumcircle. Using Ptolemy and Pythagoras, we get the equation $d^3 - d(a^2 + b^2 + c^2) - 2abc = 0$; plugging in: $d^3 - 5d - 2\sqrt{3} = 0$. Taking $d = d\sqrt{3}$ gives $3d^3 - 5d - 2 = 0$ which has root -1, so the original has root $-\sqrt{3}$; factoring gives the only positive value, $d = \frac{\sqrt{3}+\sqrt{11}}{2}$ for a final answer $\boxed{16}$.

6. Let two ants stand on the perimeter of a regular 2019-gon of unit side length. One of them stands on a vertex and the other one is on the midpoint of the opposite side. They start walking along the perimeter at the same speed counterclockwise. The locus of their midpoints traces out a figure P in the plane with N corners. Let the area enclosed by convex hull of P be $\frac{A}{B}\frac{\sin^m\left(\frac{\pi}{4038}\right)}{\tan\left(\frac{\pi}{2019}\right)}$, where A and B are coprime positive integers, and m is the smallest possible positive integer such that this formula holds. Find $A + B + m + N$.

   Proposed by Jackson Danger Blitz.

   Answer: $\boxed{6065}$.

   Solution: The area of regular a n-gon with circumradius R is $nR^2\sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right)$. The circumradius of the starting n-gon is $R_0 = \frac{1}{2\sin\left(\frac{\pi}{n}\right)}$.

   The locus consists of a 2019-star, which has 2019 isosceles triangles affixed to a smaller 2019-gon, for 4038 total corners.

   The convex hull of this star is a 2019-gon. We can find it's circumradius by noting that its vertices are precisely the points which are midpoints of a segment connecting a vertex of the 2019-gon and the midpoint of the opposite side.

   Then a simple calucation gives that $R = \frac{R_0\left(1-\cos\left(\frac{\pi}{2019}\right)\right)}{2}$. Plugging this into the formula we get that the area of the convex hull is $\frac{2019}{4}\frac{\sin^4\left(\frac{\pi}{4038}\right)}{\tan\left(\frac{\pi}{2019}\right)}$. Furthermore, $\sin\left(\frac{\pi}{4038}\right)$ is not rational when taken to the powers $1, 2, 3$. Hence the $A = 2019, B = 4, m = 4, N = 4038$, for a final answer of 6065.

Note: We also accepted the interpretation of a corner to not include the self-intersections. This gives us that $N = 2019$. Following the above logic yields the answer of 4046, which we also accepted.

7. Let $ABCD$ be a trapezoid such that $AB \parallel CD$ and let $P = AC \cap BD$, $AB = 21$, $CD = 7$, $AD = 13$, $[ABCD] = 168$. Let the line parallel to AB trough P intersect circumcircle of BCP in X. Circumcircles of BCP and APD intersect at P, Y. Let $XY \cap BC = Z$. If $\angle ADC$ is obtuse, then $BZ = \frac{a}{b}$, where a, b are coprime positive integers. Compute $a + b$.

   Proposed by Aleksa Milojević. Solution by Igor Medvedev.

   Answer: $\boxed{17}$

   Solution: The heigh of the trapezoid is $\frac{168}{14} = 12$. By using Pythagoras theorem we find $BC = 15$. Now we claim $BZ = CZ$. With this $BZ = \frac{15}{2}$, so the answer would be 17.

   Let M be the point of intersection of circumcircle of ΔAPD and XP other than P. We have $\angle DYM = \angle DPM = \angle BPX = \angle BYX$, and similarly $\angle MYA = \angle MPA = \angle XPC = \angle XYC$. Furthermore $\angle ADP = \angle APY = \angle PCB + \angle CBP - \angle YPB = \angle YCB + \angle CBY - \angle YPB = \angle CBY$. Hence triangles ΔYDA and ΔYBC are similar. The unique spiral similarity between these two triangles has center Y, call it $\Phi : \Delta YBC \mapsto \Delta YDA$.

   Let N be the point of intersection of AD and YM. Then $\Phi(Z) = N$, because $\angle BYZ = \angle BYX = \angle DYM = \angle DYN$. This means that $\frac{YZ}{YX} = \frac{YN}{YM}$. This implies that $NZ||MX$. From this we get $\frac{CZ}{ZB} = \frac{DN}{NA}$. On the other hand, $\Phi(Z) = N$, so $\frac{BZ}{ZC} = \frac{DN}{NA}$, hence $\frac{BZ}{ZC} = \frac{CZ}{ZB}$, hence $CZ = BZ$.

8. Let γ and Γ be two circles such that γ is internally tangent to Γ at a point X. Let P be a point on the common tangent of γ and Γ and Y be the point on γ other than X such that PY is tangent to γ at Y. Let PY intersect Γ at A and B, such that A is in between P and B and let the tangents to Γ at A and B intersect at C. CX intersects Γ again at Z and ZY intersects Γ again at Q. If $AQ = 6$, $AB = 10$ and $\frac{AX}{XB} = \frac{1}{4}$. The length of $QZ = \frac{p}{q}\sqrt{r}$, where p and q are coprime positive integers, and r is square free positive integer. Find $p + q + r$.

   Proposed by Mel Shu. Solution by Igor Medvedev.

   Answer: $\boxed{28}$

   Solution: P lies on the polar of C so CZ must be the polar of P so PZ is a tangent to Γ. In particular, $PZ = PY$ so there exists a circle γ' tangent to PY and PZ at Y and Z respectively. Then the homothety centered at Z which takes γ' to Γ takes Y to Q, so Q is the midpoint of arc AQB. It follows that $AQ = BQ$. Similarly XY is a bisector of angle $\angle AXB$.

   Then $\frac{AX}{XB} = \frac{BY}{YA}$ by the angle bisector theorem. Hence $BY = 2$, $AY = 8$. By Stewart's theorem applied to ABQ we have $QY^2 = AQ^2 - BY \cdot YA = 20$. Then by power of a point, $YZ = \frac{16}{\sqrt{20}}$. Then $QZ = \frac{18}{5}\sqrt{5}$.