P U M $\therefore$ C

Geometry A

1. A right cone in xyz-space has its apex at $(0,0,0)$, and the endpoints of a diameter on its base are $(12,13,-9)$ and $(12,-5,15)$. The volume of the cone can be expressed as $a\pi$. What is a?

2. Let ΔABC be a triangle with circumcenter O and orthocenter H. Let D be a point on the circumcircle of ABC such that $AD \perp BC$. Suppose that $AB = 6$, $DB = 2$, and the ratio $\frac{\text{area}(\Delta ABC)}{\text{area}(\Delta HBC)} = 5$. Then, if OA is the length of the circumradius, then OA^2 can be written in the form $\frac{m}{n}$, where m, n are relatively prime positive integers. Compute $m + n$.

3. Suppose we choose two numbers $x, y \in [0,1]$ uniformly at random. If the probability that the circle with center (x, y) and radius $|x - y|$ lies entirely within the unit square $[0,1] \times [0,1]$ is written as $\frac{p}{q}$ with p and q relatively prime nonnegative integers, then what is $p^2 + q^2$?

4. Let $BC = 6, BX = 3, CX = 5$, and let F be the midpoint of BC. Let $AX \perp BC$ and $AF = \sqrt{247}$. If AC is of the form $\sqrt{b}$ and AB is of the form $\sqrt{c}$ where b and c are nonnegative integers, find $2c + 3b$.

5. Let Γ be a circle with center A, radius 1 and diameter BX. Let Ω be a circle with center C, radius 1 and diameter DY, where X and Y are on the same side of AC. Γ meets Ω at two points, one of which is Z. The lines tangent to Γ and Ω that pass through Z cut out a sector of the plane containing no part of either circle and with angle $60°$. If $\angle XYC = \angle CAB$ and $\angle XCD = 90°$, then the length of XY can be written in the form $\frac{\sqrt{a}+\sqrt{b}}{c}$ for integers a, b, c where $\gcd(a, b, c) = 1$. Find $a + b + c$.

6. Let two ants stand on the perimeter of a regular 2019-gon of unit side length. One of them stands on a vertex and the other one is on the midpoint of the opposite side. They start walking along the perimeter at the same speed counterclockwise. The locus of their midpoints traces out a figure P in the plane with N corners. Let the area enclosed by convex hull of P be $\frac{A}{B}\frac{\sin^m\left(\frac{\pi}{4038}\right)}{\tan\left(\frac{\pi}{2019}\right)}$, where A and B are coprime positive integers, and m is the smallest possible positive integer such that this formula holds. Find $A + B + m + N$.

   Note: The *convex hull* of a figure P is the convex polygon of smallest area which contains P.

7. Let $ABCD$ be a trapezoid such that $AB \parallel CD$ and let $P = AC \cap BD$, $AB = 21$, $CD = 7$, $AD = 13$, $[ABCD] = 168$. Let the line parallel to AB trough P intersect circumcircle of BCP in X. Circumcircles of BCP and APD intersect at P, Y. Let $XY \cap BC = Z$. If $\angle ADC$ is obtuse, then $BZ = \frac{a}{b}$, where a, b are coprime positive integers. Compute $a + b$.

8. Let γ and Γ be two circles such that γ is internally tangent to Γ at a point X. Let P be a point on the common tangent of γ and Γ and Y be the point on γ other than X such that PY is tangent to γ at Y. Let PY intersect Γ at A and B, such that A is in between P and B and let the tangents to Γ at A and B intersect at C. CX intersects Γ again at Z and ZY intersects Γ again at Q. If $AQ = 6$, $AB = 10$ and $\frac{AX}{XB} = \frac{1}{4}$. The length of $QZ = \frac{p}{q}\sqrt{r}$, where p and q are coprime positive integers, and r is square free positive integer. Find $p + q + r$.