Individual Finals A

1. Given the graph G and cycle C in it, we can perform the following operation: add another vertex v to the graph, connect it to all vertices in C and erase all the edges from C. Prove that we cannot perform the operation indefinitely on a given graph.

   Solution: The number of edges stays constant, the graph stays connected, the number of vertices increases: at some point, we will have $|E|+1=|V|$. Then, our graph is a tree without cycles. The process stops then.

2. Prove that for every positive integer m, every prime p and every positive integer $j \le p^{m-1}$, p^m divides $\binom{p^m}{pj} - \binom{p^{m-1}}{j}$.

   Solution:

$$\binom{p^m}{pj} = \binom{p^{m-1}}{j} \prod_{i=1}^{pj-1} \left(\frac{p^m - i}{i}\right) =$$
$$= \binom{p^{m-1}}{j} \prod_{i=1}^{j-1} \left(\frac{p^{m-1} - i}{i}\right) \cdot \prod_{i \in I} \left(\frac{p^m - i}{i}\right),$$

   Where $I = \{i \in \mathbb{N} | 0 < i < pj,\ p \nmid i\}$. The second term cancels because all of the terms in both numerator and denominator are not divisible by p. The first term is precisely $\binom{p^{m-1}}{j}$.

   There's an even number of terms in the second product whenever $j \cdot (p-1)$ is even; then we can pair up the ones which evaluate to $(-1)^{j(p-1)} \pmod{p^m}$. In those cases, we are done, since $(-1)^{j(p-1)} = 1$. When j is odd and $p = 2$, then the binomial coefficients are negatives of each other mod 2^m, but we are still done because both expressions are divisible by 2^{m-1} (we can see this because the first term is the product of $\frac{2^{m-1}}{j}$ and an integer).

   Proposed by Alec Leng. Solution by Zhuo Qun Song.

3. Let $ABCDEF$ be a convex hexagon with area S such that $AB \parallel DE, BC \parallel EF, CD \parallel FA$ holds, and whose all angles are obtuse and opposite sides are not the same length. Prove that the following inequality holds: $A_{ABC} + A_{BCD} + A_{CDE} + A_{DEF} + A_{EFA} + A_{FAB} < S$, where A_{XYZ} is the area of triangle XYZ.

   Solution: Notice that out of two opposite sides one is always longer than the other one. We can label each side "red" or "blue" based on whether it's longer or shorter than the opposite side, respectively. We claim that there are no two adjacent red sides. For the sake of contradiction, assume the contrary that AB and BC are both red. Exactly one of the sides AF and CD is red (the two are opposite sides so one is blue and one is red). Without loss of generality assume that AF is red. Denote F' and C' the orthogonal projections of F and C to AB. Similarly, denote F'' and C'' the orthogonal projections of F and C to DE. Notice that $F'C' = F''C''$ (these four points form a rectangle). $\Delta AFF' \sim DCC''$ since the corresponding sides are parallel, hence $AF' > DF$" and $AF > DC$. Similarly, $BC' > EF$", so we have $F'C' = F'A + AB + BC' > F"D + DE + EF" = F"C" = F'C'$, which is absurd. Hence the red and blue sides alternate.

   Without loss of generality let AB be a blue side. Let k, l, m be lines through A, C, E parallel to BC, DE, FA respectively. Let $k \cap l = X, l \cap m = Y, , m \cap k = Z$. Since $AZEF$ and $ABCX$ are parallelograms we have that $AZ = EF < BC = AX$, so the point Z is in between A and X. Similarly, X is in between C, Y and Y is in between E, Z. Then the following holds $S =$

$A_{ABCX}+A_{CDEY}+A_{EFAZ}+A_{XYZ} > A_{ABCX}+A_{CDEY}+A_{EFAZ} = 2(A_{ABC}+A_{CDE}+A_{EFA})$. Similarly we can prove that $S > 2(A_{BCD}+A_{DEF}+A_{FAB})$. Adding up the last two inequalities, we get that $2S > 2(A_{ABC} + A_{BCD} + A_{CDE} + A_{DEF} + A_{EFA} + A_{FAB})$, which means that $S > A_{ABC} + A_{BCD} + A_{CDE} + A_{DEF} + A_{EFA} + A_{FAB}$.