P U M ∴ C

Number Theory A

1. The least common multiple of two positive integers a and b is $2^5 \times 3^5$. How many such ordered pairs (a, b) are there?

2. Let f be a function over the natural numbers so that

   (a) $f(1) = 1$

   (b) If $n = p_1^{e_1} ... p_k^{e_k}$ where $p_1, \cdots, p_k$ are distinct primes, and $e_1, \cdots e_k$ are non-negative integers, then $f(n) = (-1)^{e_1 + .. + e_k}$.

   Find $\sum_{i=1}^{2019} \sum_{d|i} f(d)$.

3. Consider the first set of 38 consecutive positive integers who all have sum of their digits not divisible by 11. Find the smallest integer in this set.

4. For a positive integer n, let $f(n) = \sum_{i=1}^{n} \lfloor \log_2 n \rfloor$. Find the largest $n < 2018$ such that $n \mid f(n)$.

5. Call a positive integer n *compact* if for any infinite sequence of distinct primes $p_1, p_2, \ldots$ there exists a finite subsequence of n primes $p_{x_1}, p_{x_2}, \ldots p_{x_n}$ (where the x_i are distinct) such that

$$p_{x_1} p_{x_2} \cdots p_{x_n} \equiv 1 \pmod{2019}$$

   Find the sum of all *compact* numbers less than $2 \cdot 2019$.

6. Let $p, q \le 200$ be prime numbers such that $\frac{q^p - 1}{p}$ is a square. Find the sum of $p + q$ over all such pairs.

7. Let $f(x)$ be the nonnegative remainder when x is divided by the prime $p = 1297$. Let $g(x)$ be the largest possible value of $f(-p_1) + f(-p_2) + \ldots + f(-p_m)$ over all sets $\{p_1, \ldots, p_m\}$ where p_k are primes such that for all $1 \le i < j \le m$ we have $p \nmid (p_i^2 - p_j^2)$, and

$$p \nmid \sigma((p_1 \times \ldots \times p_m)^{x-1}),$$

   where $\sigma(x)$ is the sum of the (distinct, positive, not necessarily proper) divisors of x. Find

$$\sum_{k=1}^{(p+1)/2} \left(g(p - 2k + 3) - g(p + 2k + 1) \right).$$

8. The integer 107 is a prime number. Let $p = 107$. For an integer a such that $p \nmid a$ let a^{-1} be the unique integer $0 \le a^{-1} \le p^2 - 1$ such that $p^2 | aa^{-1} - 1$. Find the number of positive integers b, $1 \le b \le \frac{p^2-1}{2}$ such that there exists an integer a, $0 \le a \le p^2 - 1$ such that $p^2 | b^2 - (a + a^{-1})$.