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Number Theory A Solutions

1. The least common multiple of two positive integers a and b is $2^5 \times 3^5$. How many such ordered pairs (a, b) are there?

   Proposed by: Rahul Saha

   Answer: $\boxed{121}$

   Looking at each prime, there are 11 choices, so the answer is 11^2.

2. Let f be a function over the natural numbers so that

   1. $f(1) = 1$

   2. If $n = p_1^{e_1}...p_k^{e_k}$ where $p_1, \cdots, p_k$ are distinct primes, and $e_1, \cdots e_k$ are non-negative integers, then $f(n) = (-1)^{e_1+..+e_k}$.

   Find $\sum_{i=1}^{2019} \sum_{d|i} f(d)$.

   Proposed by: Marko Medvedev

   Answer: $\boxed{44}$

   Since the function is completely multiplicative, $\sum_{d|i} f(d)$ is given by product of $\dfrac{f(p_k)^{x_k+1} - 1}{f(p_k) - 1}$ which is 0 if x_k is odd and 1 if x_k is even (recall that $f(p) = -1$ for all primes p). Therefore the required sum evaluates to the number of squares less than 2019, which is 44.

3. Consider the first set of 38 consecutive positive integers who all have sum of their digits not divisible by 11. Find the smallest integer in this set.

   Proposed by: Marko Medvedev

   Answer: $\boxed{999981}$

   Consider first the last two digits. Note that if we don't go past a multiple of 100, then we will have a string of at least 12 consecutive sums of digits since we will have a number ending in zero such that 29 plus that number has sum of digits 11 more than that number. Note that if we go up to at least 19 mod 100 then we will have 11 consecutive sums, and if we go down to at most 80 then we will have 11 consecutive sums, so we must have the range from $100x + 81$ to $100x + 118$. Then we must have the sum of digits of $100x + 100$ must have sum 1 mod 11, so $x + 1$ has sum of digits 1 mod 11, and $100x + 81$ must have sum 1 mod 11 so x has sum 3 mod 11. Thus when we add 1 to x we have to increase digitsum by 9 mod 11. Note that x must end in some number of nines. If it ends in k nines, then we increase by $1 - 9k$ Thus $2k + 1 = 9 \pmod{11}$ so $k = 4$ so the smallest x is 9999 and our answer is 999981.

4. For a positive integer n, let $f(n) = \sum_{i=1}^{n} \lfloor \log_2 i \rfloor$. Find the largest $n < 2018$ such that $n \mid f(n)$.

   Proposed by: Eric Neyman

   Answer: $\boxed{1013}$

   First note that

$$f(2^{r+1} - 1) = \sum_{k=0}^{r} k \cdot 2^k = \sum_{i=1}^{r} \sum_{j=i}^{r} 2^j = \sum_{i=1}^{r} (2^{r+1} - 2^i) = (r-1)2^{r+1} + 2.$$

   Thus, if we write $n = 2^{r+1} - 1 + m$, where $0 \le m \le 2^{r+1}$, we have

$$f(n) = (r-1)2^{r+1} + 2 + m(r+1).$$

Thus, the condition $n \mid f(n)$ is equivalent (after subtracting $(r-1)n$ from $f(n)$) to

$$2^{r+1} - 1 + m \mid 2 + m(r+1) + r - 1 - m(r-1) = 2m + r + 1.$$

Now, the right-hand side is more than zero times the left-hand side but more than twice the left-hand side, so $n \mid f(n)$ if and only if $2^{r+1} - 1 + m = 2m + r + 1$, i.e. $m = 2^{r+1} - r - 2$, so $n = 2^{r+2} - r - 3$.

The largest such value that is less than 2018 is $2^{10} - 8 - 3 = 1013$.

5. Call a positive integer n *compact* if for any infinite sequence of distinct primes $p_1, p_2, \ldots$ there exists a finite subsequence of n primes $p_{x_1}, p_{x_2}, \ldots p_{x_n}$ (where the x_i are distinct) such that

$$p_{x_1} p_{x_2} \cdots p_{x_n} \equiv 1 \pmod{2019}$$

Find the sum of all *compact* numbers less than $2 \cdot 2019$.

Proposed by: Rahul Saha

Answer: $\boxed{14112}$

Claim 1: Let n be a compact number. Then we must have $a^n \equiv 1 \pmod{2019}$ for all $(a, 2019) = 1$.

Proof: By Dirichlet's theorem on arithmetic progressions, we can find infinitely many primes $p \equiv a \pmod{2019}$. Letting our sequence be composed only of these primes, we must have $a^n \equiv 1 \pmod{2019}$.

Claim 2: If $a^n \equiv 1 \pmod{2019}$ for all $(a, 2019) = 1$, then n is a compact number.

Proof: Note that by taking all large enough primes in our sequence, we can assume $(p_i, 2019) = 1$. But some residue a mod 2019 must appear infinitely many times, which gives us $a^n \equiv 1 \pmod{2019}$, as desired.

Claim 3: Let n be the minimal compact number. Then all compact numbers are multiples of n, and conversely any multiple of n is a good number.

Proof: Let N be another compact number, and suppose $N = nq + r$, but then we have $a^N \equiv a^r \equiv 1$ which would make r the minimal good number, a contradiction unless $r = 0$. The other direction is trivial.

Claim 4: The minimal compact number is 672.

Proof: Let x and y be primitive roots modulo 3 and 673. Then the order of xy is $\frac{2 \cdot 672}{(2,672)} = 672$, so the minimal compact number is at least 672. Note, $a^{672} \equiv 1 \pmod 3$ and $a^{672} \equiv 1 \pmod{673}$ therefore $a^{672} \equiv 1 \pmod{2019}$ for all $(a, 2019) = 1$. Therefore the minimal compact number is 672.

Therefore, the sum is $672 \cdot (1 + 2 + 3 + 4 + 5 + 6) = 672 \cdot 21 = 14112$.

6. Let $p, q \le 200$ be prime numbers such that $\frac{q^p - 1}{p}$ is a square. Find the sum of $p + q$ over all such pairs.

Proposed by: Marko Medvedev

Answer: $\boxed{24}$

We have that $p | q^p - 1$, hence $p | q - 1$ by Fermat's small theorem. Now suppose that p is odd. Then we have that $v_p(q^p - 1) = v_p(q - 1) + 1$, so we have that $p || \frac{q^p - 1}{q - 1}$ and furthermore that $q - 1$ and $\frac{q^p - 1}{p(q-1)}$ are coprime, and hence squares. Then $q - 1$ is square, and is divisible by p so it's $q = (pm)^2 + 1$ for some integer m. Furthermore since p is odd, $q > 2$ hence also odd. Then q is of the form $q = (2pm)^2 + 1$ for some integer m. Now since we have $q \le 200$ we can check all cases directly (there's three of them), and get that there are no solutions here. Now suppose that $p = 2$. Hence $q^2 = 2x^2 + 1$. Since $\frac{1}{2}(q-1)(q+1) = x^2$, and $(q-1, q+1) = 2$, we know that x is even. If x is divisible by 4 then $q \equiv 1 \pmod{16}$, so $q \equiv 1 \pmod 8$. Furthermore by looking at modulo 32 it's clear that $q \equiv 1 \pmod{16}$. This eases the search a lot and the only answer here is $q = 17$ and we can check that his indeed works. Now if $x \equiv 2 \pmod 4$, then $q \equiv 3 \pmod 4$, since the only odd integers dividing $x^2 + 1$ are of the form $4k + 1$. Then looking at $\pmod{16}$ gives $q \equiv 3 \pmod{16}$. Again the search is greatly reduced and we get that the only solution is $q = 3$. In total the solutions are $(p, q) = (2, 3), (2, 17)$.

7. Let $f(x)$ be the nonnegative remainder when x is divided by the prime $p = 1297$. Let $g(x)$ be the largest possible value of $f(-p_1) + f(-p_2) + \ldots + f(-p_m)$ over all sets $\{p_1, \ldots, p_m\}$ where p_k are primes such that for all $1 \le i < j \le m$ we have $p \nmid (p_i^2 - p_j^2)$, and

$$p \nmid \sigma((p_1 \times \ldots \times p_m)^{x-1}),$$

where $\sigma(x)$ is the sum of the (distinct, positive, not necessarily proper) divisors of x. Find

$$\sum_{k=1}^{(p+1)/2} \left(g(p - 2k + 3) - g(p + 2k + 1)\right).$$

Proposed by: Michael Gintz

Answer: $\boxed{2557}$

By dirichlet's theorem, we can find a prime with any value mod p. Now note that σ is the product of $(p_k^x - 1)/(p_k - 1)$. If p_k is 1 mod p, then the value it multiplies is not 0 mod p unless x is 0 mod p. Thus we have values 1 mod p here except in $g(2p)$. Thus for 2 to $p - 1$ we can simply consider whether p_k^x is 1 mod p, and then take the max of p_k and $p - p_k$. Define h as g but the p_k cannot be 1 mod p.

Note for that we can arbitrarily choose some primitive root r, write every number from 2 to $p - 2$ as r^k, and then to see whether we can include $f(r)$ in $g(x)$ we simply see if $(p - 1) \nmid xk$. Then we have that $h(x) = h(x + p - 1)$ and $h(x) = h(p - 1 - x)$, and thus we are looking for

$$h(2) + \ldots + h(p + 1) - h(p + 3) - \ldots - h(2p + 2) + (p - 1)$$

$$= 2h(p + 1) - h(2p) - h(2p + 2) + (p - 1)$$

$$= h(2) - h(4) + (p - 1)$$

where the $(p - 1)$ comes from the fact that $g(2p)$ cannot include $p_k \equiv \pm 1 \pmod p$. Note that $h(2)$ can include everything whose square is not 1 mod p, which is everything from $(p + 1)/2$ to $p - 2$. Then note that $h(4)$ contains everything whose 4th power is not 1 mod p. Note that $1296 = 6^4$, so 36 is a 4th root. Thus this is everything from $(p + 1)/2$ to $p - 2$ except $p - 36$. Thus $h(2) - h(4) = (p - 36)$ and our answer is $2p - 37 = 2557$.

8. The number 107 is a prime number. Let $p = 107$. For a number a such that $p \nmid a$ let a^{-1} be the unique number $0 \le a^{-1} \le p^2 - 1$ such that $p^2 | aa^{-1} - 1$. Find the number of positive integers b, $1 \le b \le \frac{p^2 - 1}{2}$ such that there exists a number a, $0 \le a \le p^2 - 1$ such that $p^2 | b^2 - (a + a^{-1})$.

Proposed by: Igor Medvedev

Answer: $\boxed{2783 \ (\frac{p^2 - 3p + 4}{4})}$

Solutions: We work in $\pmod{p^2}$. First note that for $4 \mid p-3$, -1 is not a quadratic residue mod p^2. Then note that for $p \nmid x$, x is a quadratic residue $\pmod{p^2}$ iff $-x$ is not a quadratic residue (mod p^2). Now we will count the number of values that $a+a^{-1}$ takes in $\{0,1,2,\ldots,p^2-1\}$ takes as a ranges over $0,1,\ldots,p^2-1$. Suppose that for numbers x,y we have that $x+x^{-1}=y+y^{-1}$. This is equivalent to $p^2 \mid (xy-1)(x-y)$. For $x=kp+1$ the value is 2. Similarly for $y=kp-1$, the value is -2, and exactly one of these two is a quadratic residue. For $x \neq \pm 1 \pmod{p}$, there exists exactly one $y=x^{-1}, y \neq x$ such that $x+x^{-1}=y+y^{-1}$, since we have to have either $p^2 \mid xy-1$ or $p^2 \mid x-y$. Now for $x \neq \pm 1 \pmod{1}$, we have that $x+x^{-1}=-(-x+(-x)^{-1})$, so exactly one of these is a quadratic residue. Then for each of the p^2-3p problems which don't give $-1,0,1 \pmod{p}$ we pair up $x+x^{-1}$ with the corresponding $y+y^{-1}$ and $x+x^{-1}$ with $-(-x+(-x)^{-1})$. Exactly one of these values is counted, so this adds $\frac{p^2-3p}{4}$. To this we add one for 2 or -2. The total number is $\frac{p^2-3p+4}{4}$.