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Number Theory B Solutions

1. The product of the positive factors of a positive integer n is 8000. What is n?

   Proposed by: Jacob Wachspress, Nathan Bergman

   Answer: $\boxed{20}$

   20 has 6 factors, 3 pairs, this is $20^3 = 8000$.

2. The least common multiple of two positive integers a and b is $2^5 \times 3^5$. How many such ordered pairs (a, b) are there?

   Proposed by: Rahul Saha

   Answer: $\boxed{121}$

   Looking at each prime, there are 11 choices, so the answer is 11^2.

3. Let f be a function over the natural numbers so that

   1. $f(1) = 1$

   2. If $n = p_1^{e_1}...p_k^{e_k}$ where $p_1, \cdots, p_k$ are distinct primes, and $e_1, \cdots e_k$ are non-negative integers, then $f(n) = (-1)^{e_1+..+e_k}$.

   Find $\sum_{i=1}^{2019} \sum_{d|i} f(d)$.

   Proposed by: Marko Medvedev

   Answer: $\boxed{44}$

   Since the function is completely multiplicative, $\sum_{d|i} f(d)$ is given by product of $\dfrac{f(p_k)^{x_k+1} - 1}{f(p_k) - 1}$ which is 0 if x_k is odd and 1 if x_k is even (recall that $f(p) = -1$ for all primes p). Therefore the required sum evaluates to the number of squares less than 2019, which is 44.

4. Let n be the smallest positive integer which can be expressed as a sum of multiple (at least two) consecutive integers in precisely 2019 ways. Then n is the product of k not necessarily distinct primes. Find k.

   Proposed by: Oliver Thakar

   Answer: $\boxed{105}$

   n can be written as a sum of $2k + 1$ consecutive integers if and only if $2k + 1$ is a divisor of n, for letting x be the integer in the center of the sum, then $n = (x - k) + \ldots + x + \ldots + (x + k) = (2k + 1)x$. Hence, the number of odd divisors of n minus one (1 is an odd divisor of n but does not correspond to a sum of at least two consecutive integers) equals the number of ways that n can be written as a sum of an odd number of consecutive integers.

   There is a bijection between each writing of n as a sum of consecutive even integers and the odd divisors of n.

   Letting $2k$ be an even integer, and x being the center of the sum (so that x is some integer plus $\frac{1}{2}$,) then:

   $$n = (x - \frac{1}{2} - k) + \ldots + (x - \frac{1}{2}) + (x + \frac{1}{2}) + \ldots + (x + \frac{1}{2} + k) = kx.$$

   Thus, we know that the factor of 2 in k must be one more than the factor of 2 in n, and that $2k$ divides n. For each odd divisor of n, there is one such k which is a power of 2 times that divisor, and vice versa.

Thus, the total number of ways to write n as the sum of multiple consecutive integers is 2 times the number of odd divisors minus 1.

The number of odd divisors of n must be 1010, then, which factors into $1010 = 2 * 5 * 101$.

Now, there are three prime factors of 1010, so there can be up to 3 odd prime factors of n. (And, the smallest n will have no even prime factors.)

If there is only one prime factor, then $n = 3^{1009}$, for n must be the smallest.

If there are two prime factors, then there are three possibilities: $n = p^{100}q^9$ or $n = p^{201}q^4$ or $n = p^{504}q$. Clearly, in all three cases, the smallest n results in $p = 3$ and $q = 5$. Of all three cases, the smallest is $n = 3^{100}5^9$, for $3^101 > 3^8 > 5^5$. Similarly, $n = 3^{100}5^9$ is smaller than $n = 3^{1009}$, for $3^{909} > 3^{14} > 5^9$.

Now, suppose there are three prime factors. Here, $n = 3^{100} \cdot 5^4 \cdot 7$. Since $5^4 \cdot 7 < 5^9$, then this solution is the smallest possible.

Thus, the answer is $n = 3^{100} \cdot 5^4 \cdot 7$.

5. Consider the first set of 38 consecutive positive integers who all have sum of their digits not divisible by 11. Find the smallest integer in this set.

Proposed by: Marko Medvedev

Answer: $\boxed{999981}$

Consider first the last two digits. Note that if we don't go past a multiple of 100, then we will have a string of at least 12 consecutive sums of digits since we will have a number ending in zero such that 29 plus that number has sum of digits 11 more than that number. Note that if we go up to at least 19 mod 100 then we will have 11 consecutive sums, and if we go down to at most 80 then we will have 11 consecutive sums, so we must have the range from $100x + 81$ to $100x + 118$. Then we must have the sum of digits of $100x + 100$ must have sum 1 mod 11, so $x + 1$ has sum of digits 1 mod 11, and $100x + 81$ must have sum 1 mod 11 so x has sum 3 mod 11. Thus when we add 1 to x we have to increase digitsum by 9 mod 11. Note that x must end in some number of nines. If it ends in k nines, then we increase by $1 - 9k$ Thus $2k + 1 = 9 \pmod{11}$ so $k = 4$ so the smallest x is 9999 and our answer is 999981.

6. Let f be a polynomial with integer coefficients of degree 2019 such that the following conditions are satisfied:

1. For all integers n, $f(n) + f(-n) = 2$.

2. $101^2 \mid f(0) + f(1) + f(2) + \cdots + f(100)$. Compute the remainder when $f(101)$ is divided by 101^2.

Proposed by: Matthew Kendall

Answer: $\boxed{203}$

We use this fact: For nonnegative integer k and prime $p > 2$,

$$p^2 \mid 1^{2k+1} + 2^{2k+1} + \cdots + (p-1)^{2k+1}.$$

This comes from $j^{2k+1} + (p-j)^{2k+1} \equiv (2k+1)jp \pmod{p^2}$ and summing over all j.

Let $p = 101$. Plugging $n = 0$ into 1 gives $f(0) = 1$. Since $\deg f = 2019$, we can write $f(n) = 1 + an + g(n)$ where g is odd and all of its terms are of degree at least 3. Now using fact 2, $p^2 \mid g(0) + g(1) + g(2) + \cdots + g(p)$. This means

$$f(0) + f(1) + f(2) + \cdots + f(100) \equiv p + a(1 + \cdots + (p-1)). \pmod{p^2}$$

So $2p + ap(p-1) \equiv 0 \pmod{p^2}$ or $ap \equiv 2p \pmod{p^2}$. Hence, $f(p) = 1 + ap \equiv 2p + 1 \pmod{p^2}$. Plugging in $p = 101$ gives $f(101) \equiv 203 \pmod{101^2}$.

7. For a positive integer n, let $f(n) = \sum_{i=1}^{n} \lfloor \log_2 n \rfloor$. Find the largest $n < 2018$ such that $n \mid f(n)$.

   Proposed by: Eric Neyman

   Answer: $\boxed{1013}$

   First note that

$$f(2^{r+1}-1) = \sum_{k=0}^{r} k \cdot 2^k = \sum_{i=1}^{r}\sum_{j=i}^{r} 2^j = \sum_{i=1}^{r}(2^{r+1}-2^i) = (r-1)2^{r+1}+2.$$

   Thus, if we write $n = 2^{r+1}-1+m$, where $0 \le m \le 2^{r+1}$, we have

$$f(n) = (r-1)2^{r+1} + 2 + m(r+1).$$

   Thus, the condition $n \mid f(n)$ is equivalent (after subtracting $(r-1)n$ from $f(n)$) to

$$2^{r+1} - 1 + m \mid 2 + m(r+1) + r - 1 - m(r-1) = 2m + r + 1.$$

   Now, the right-hand side is more than zero times the left-hand side but more than twice the left-hand side, so $n \mid f(n)$ if and only if $2^{r+1} - 1 + m = 2m + r + 1$, i.e. $m = 2^{r+1} - r - 2$, so $n = 2^{r+2} - r - 3$.

   The largest such value that is less than 2018 is $2^{10} - 8 - 3 = 1013$.

8. Call a positive integer n *compact* if for any infinite sequence of distinct primes $p_1, p_2, \ldots$ there exists a finite subsequence of n primes $p_{x_1}, p_{x_2}, \ldots p_{x_n}$ (where the x_i are distinct) such that

$$p_{x_1}p_{x_2}\cdots p_{x_n} \equiv 1 \pmod{2019}$$

   Find the sum of all *compact* numbers less than $2 \cdot 2019$.

   Proposed by: Rahul Saha

   Answer: $\boxed{14112}$

   Claim 1: Let n be a compact number. Then we must have $a^n \equiv 1 \pmod{2019}$ for all $(a, 2019) = 1$.

   Proof: By Drichlet's theorem on arithmetic progressions, we can find infinitely many primes $p \equiv a \pmod{2019}$. Letting our sequence be composed only of these primes, we must have $a^n \equiv 1 \pmod{2019}$.

   Claim 2: If $a^n \equiv 1 \pmod{2019}$ for all $(a, 2019) = 1$, then n is a compact number.

   Proof: Note that by taking all large enough primes in our sequence, we can assume $(p_i, 2019) = 1$. But some residue a mod 2019 must appear infinitely many times, which gives us $a^n \equiv 1 \pmod{2019}$, as desired.

   Claim 3: Let n be the minimal compact number. Then all compact numbers are multiples of n, and conversely any multiple of n is a good number.

   Proof: Let N be another compact number, and suppose $N = nq + r$, but then we have $a^N \equiv a^r \equiv 1$ which would make r the minimal good number, a contradiction unless $r = 0$. The other direction is trivial.

   Claim 4: The minimal compact number is 672.

   Proof: Let x and y be primitive roots modulo 3 and 673. Then the order of xy is $\frac{2 \cdot 672}{(2,672)} = 672$, so the minimal compact number is at least 672. Note, $a^{672} \equiv 1 \pmod 3$ and $a^{672} \equiv 1 \pmod{673}$ therefore $a^{672} \equiv 1 \pmod{2019}$ for all $(a, 2019) = 1$. Therefore the minimal compact number is 672.

   Therefore, the sum is $672 \cdot (1+2+3+4+5+6) = 672 \cdot 21 = 14112$.