Team Round

1. Two unit squares are stacked on top of one another to form a 1×2 rectangle. Each of the seven edges is colored either red or blue. How many ways are there to color the edges in this way such that there is exactly one path along all-blue edges from the bottom-left corner to the top-right corner?

   Proposed by: Nathan Bergman

   Answer: $\boxed{30}$

   There are four cases. First, the path that goes right then up; there are 10 ways to color this. Then the path that goes up then right; there are also 10 for this by symmetry. The path that goes up, right, up has 9 ways to be colored. Lastly, the path that goes right, up, left, up, right has 1 way. Then the answer is 30.

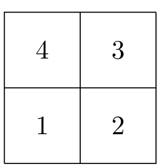
2. In a standard game of Rock–Paper–Scissors, two players repeatedly choose between rock, paper, and scissors, until they choose different options. Rock beats scissors, scissors beats paper, and paper beats rock. Nathan knows that on each turn, Richard randomly chooses paper with probability 33%, scissors with probability 44%, and rock with probability 23%. If Nathan plays optimally against Richard, the probability that Nathan wins is expressible as a/b where a and b are coprime positive integers. Find $a + b$.

   Proposed by: Nathan Bergman

   Answer: $\boxed{89}$

   Playing paper obviously sucks. Rock wins $\frac{4}{7}$ of the time. Scissors wins $\frac{33}{56}$ of the time. $\frac{33}{56}$ is best so play scissors.

3. Julia is placing identical 1-by-1 tiles on the 2-by-2 grid pictured, one piece at a time, so that every piece she places after the first is adjacent to, but not on top of, some piece she's already placed. Determine the number of ways that Julia can complete the grid.



   Proposed by: Frank Lu

   Answer: $\boxed{16}$

   Julia can choose any of the 4 pieces to place first. Next, she can choose to place any 2 of the pieces adjacent to this first piece. From here, she can place the final two pieces in any order, since both are adjacent to one of the two pieces already placed. There are 2 choices for such an order, This gives us a total of $4 * 2 * 2 = 16$ ways for Julia to fill the grid.

   Note: We also accepted the interpretation where the pieces cannot be placed vertically on top of another. The solution for this case is as follows:

   Note that in a given column, we must place the piece at the top before the piece at the bottom. We have 2 choices for the first piece, taking either the top left or top right corner. Then, if our 2nd piece fills up the top row, we have 2 more ways to fill in the grid to get 4 ways in this case. Else, our second piece fills up the column, and our last 2 pieces must be, in order, placing the top corner and bottom corner of the opposite corner. This gives us $4 + 2 = 6$ cases, which we also accepted as an answer.

4. What is the sum of the leading (first) digits of the integers from 1 to 2019 when the integers are written in base 3? Give your answer in base 10.

   Proposed by: Nathan Bergman

   Answer: $\boxed{2945}$

   For integers from 1 to 2186, there are an equal number with leading digit 1 and leading digit 2 (as 2187 is a power of 3). The integers from 2020 to 2186 all have leading digit 2, so the answer is $2186 * \frac{3}{2} - 2 * 167 = 2945$.

5. Let $f(x) = x^3 + 3x^2 + 1$. There is a unique line of the form $y = mx + b$ such that $m > 0$ and this line intersects $f(x)$ at three points, A, B, C such that $AB = BC = 2$. Find $\lfloor 100m \rfloor$.

   Proposed by: Frank Lu

   Answer: $\boxed{41}$

   Let $g(x) = mx+b$. We know that $f(x)-g(x) = (x-k)(x-k-d)(x-k+d)$ for some real numbers k, d. Expanding this out gives us that $f(x)-g(x) = x^3-3kx^2+(3k^2-d^2)x+(-k^3+kd^2)$, which means that $k = -1$. This then simplifies to $x^3+3x^2+1-mx-b = x^3+3x^2+(3-d^2)x-d^2+1$. Observe that in order for this to hold, since the RHS has a root at $x = -1$, the LHS needs to as well. Plugging in $x = -1$ here yields that $3 + m - b = 0 \implies b = m + 3$, so we have $x^3+3x^2-2-m(x+1) = (x^2+2x-2-m)(x+1) = (x+1)((x+1)^2-d^2)$. Factoring out $(x+1)$ yields $x^2+2x-2-m = (x+1)^2-d^2$, which implies that $d^2 = (m+3)$. Now, applying simple trigonometry, the distance $AB = BC$ is equal to $d\sqrt{m^2+1} = \sqrt{(m^2+1)(m+3)} = 2$. We now solve for m. Squaring both sides and moving everything to one side yields $m^3+3m^2+m-1 = 0$. Noting that $m+1$ is a factor of this, we see that we this factors to $(m^2+2m-1)(m+1) = 0$. We want positive m, so solving the quadratic and taking the positive solution yields $\sqrt{2}-1$. This yields the answer $\lfloor 100(\sqrt{2}-1) \rfloor = 41$

6. Pavel and Sara roll two, fair six-sided dice (with faces labeled from 1 to 6) but do not look at the result. A third-party observer whispers the product of the face-up numbers to Pavel and the sum of the face-up numbers to Sara.

   Pavel and Sara are perfectly rational and truth-telling, and they both know this.

   Pavel says, "With the information I have, I am unable to deduce the sum of the two numbers rolled."

   Sara responds, "Interesting! With the information I have, I am unable to deduce the product of the two numbers rolled."

   Pavel responds, "Wow! I still cannot deduce the sum. But I'm sure you know the product by now!"

   What is the product?

   Proposed by: Jacob Wachspress

   Answer: $\boxed{6}$

   The only products that may arise in multiple ways are $12 = 4 \cdot 3 = 6 \cdot 2$, $6 = 3 \cdot 2 = 6 \cdot 1$, and $4 = 2 \cdot 2 = 4 \cdot 1$. Thus, Pavel must have received one of $\{4, 6, 12\}$, or else he would have been able to deduce the two numbers and their sum.

   The possible sums for numbers with a product of 12 are 7 and 8, the possible sums for numbers with a product of 6 are 5 and 7, and possible sums for numbers with a product of 4 are 4 and 5. Only 5 and 7 appear multiple times, so Sara must have received one of these numbers; otherwise knowing that Pavel received one of $\{4, 6, 12\}$ (which she learns when he says he does not know the sum) would allow her to deduce the numbers and their product.

If Pavel does not know the sum of the numbers despite knowing the product and that the sum is 5 or 7, then the product must be 6, since of the remaining options only 6 can be written as the product of two numbers that sum to 5 and as the product of two numbers that sum to 7.

7. For all sets A of complex numbers, let $P(A)$ be the product of the elements of A. Let $S_z = \left\{1, 2, 9, 99, 999, \frac{1}{z}, \frac{1}{z^2}\right\}$, let T_z be the set of nonempty subsets of S_z (including S_z), and let $f(z) = 1 + \sum_{s \in T_z} P(s)$. Suppose $f(z) = 6125000$ for some complex number z. Compute the product of all possible values of z.

   Proposed by: Kapil Chandran

   Answer: $\boxed{48}$

   Fact: if $A = \{a_1, \ldots, a_n\}$, then $f(A) = \prod_{i=1}^{n}(a_i + 1)$. This is easily checked by expanding and observing that the monomial $\prod_{i \in I} a_i$ corresponds precisely and uniquely to the subset $\{a_i \mid i \in I\} \subset A$ for any $I \subset \{1, \ldots, n\}$.

   Write $y = \frac{1}{z}$. We wish to solve $6125000 = 2 \cdot 3 \cdot 10 \cdot 100 \cdot 1000(y+1)(y^2+1) = 6000000(y+1)(y^2+1)$. By Vieta's, the product of all possible values of y is $\frac{6125000-6000000}{6000000} = \frac{1}{48}$, so reciprocating gives 48 as the final answer.

8. The curves $y = x + 5$ and $y = x^2 - 3x$ intersect at points A and B. C is a point on the lower curve between A and B. The maximum possible area of the quadrilateral $ABCO$ can be written as $\frac{A}{B}$ for coprime A, B. Find $A + B$.

   Proposed by: Yuxi Zheng

   Answer: $\boxed{253}$

   $y = x+5$ and $y = x^2 - 3x$ intersect at points $A(-1, 4)$ and $B(5, 10)$. Segment OB is a straight line through the origin with the equation $y = 2x$. Let $C(x, x^2 - 3x)$ be a point on the curve between O and B. The vertical distance between point C and segment OB is $2x - (x^2 - 3x)$. The area of triangle OBC can be obtained by $\frac{1}{2} \times (x_B - x_O) \times (2x - (x^2 - 3x))$, where x_B and x_O denote the x coordinates of point B and point O respectively. Therefore, the area of triangle OBC equals $\frac{1}{2} \times 5 \times (2x - (x^2 - 3x))$. After completing the squre we can get that the maximum area of OBC happens when C is $(\frac{5}{2}, -\frac{5}{4})$, and the area is $\frac{125}{8}$. The intersection of AB with the y axis is $(0, 5)$, therefore the area of triangle OAB can be easily calculated as $\frac{1}{2} \times (x_B - x_A) \times 5 = 15$. Therefore, the area of OABC is simply $\frac{245}{8}$.

9. Find the integer $\sqrt[5]{55^5 + 3183^5 + 28969^5 + 85282^5}$.

   Proposed by: Jackson Blitz

   Answer: $\boxed{85359}$

   Let k be the desired integer. Then we have

$$k^5 < 85282^5 + 30000^5 < 85282^5 + 5 \cdot 100 \cdot 85282^4$$

$$< (85282 + 100)^5 = (85382)^5$$

   so 85282 is less than $k < 85382$. Taking the original equation $\mod 3$ gives $k^5 \equiv 0 \mod 3$ so $k \equiv 0 \mod 3$. Taking the original equation $\mod 5$ gives $k^5 \equiv 4 \mod 5$ so $k \equiv 4 \mod 5$.. Taking the original equation $\mod 8$ gives $k^5 \equiv 7 \mod 8$ so $k \equiv 7 \mod 8$. These bounds and equivalences imply $k = 85359$.

10. Define the unit N-hypercube to be the set of points $[0,1]^N \subset \mathbb{R}^N$. For example, the unit 0-hypercube is a point, and the unit 3-hypercube is the unit cube. Define a k-face of the unit N-hypercube to be a copy of the k-hypercube in the exterior of the N-hypercube. More formally, a k-face of the unit N-hypercube is a set of the form

$$\prod_{i=1}^{N} S_i$$

where S_i is either $\{0\}$, $\{1\}$, or $[0,1]$ for each $1 \le i \le N$, and there are exactly k indices i such that $S_i = [0,1]$.

The expected value of the dimension of a random face of the unit 8-hypercube (where the dimension of a face can be any value between 0 and N) can be written in the form $\frac{p}{q}$ where p and q are relatively prime positive integers. Find $p+q$.

Proposed by: Matt Tyler

Answer: 11

Solution by Michael Gintz

Note that there is a bijection between the faces of this hypercube and elements of $\{0,[1,0],1\}^8$ by definition. Then the dimension of a k-face is the number of $[0,1]$'s in the 8-element set corresponding to it. By linearity this is on average $8/3$.

11. The game Prongle is played with a special deck of cards: on each card is a nonempty set of distinct colors. No two cards in the deck contain the exact same set of colors. In this game, a "Prongle" is a set of at least 2 cards such that each color is on an even number of cards in the set. Let k be the maximum possible number of prongles in a set of 2019 cards. Find $\lfloor \log_2(k) \rfloor$.

Proposed by: Michael Gintz

Answer: 2007

Consider every card as a vector in the mod-2 vector space of c variables, where c is the number of colors used. If the dimension is x, then take a set of x linearly independent cards which will be our basis, and every other subset of non-basis cards will have exactly 1 subset of the basis which gives us a Prongle, so 2^{2019-x} sets will give us a solution. Now note that we cannot have 2019 cards with dimension 10, so we must have dimension at least 11, which we can do by making sure that we have 11 cards with one color each. Then there are $2^{2008}-1$ solutions, where the -1 comes from the empty set.

12. In quadrilateral $ABCD$, angles A, B, C, D form an increasing arithmetic sequence. Also, $\angle ACB = 90°$. If $CD = 14$ and the length of the altitude from C to AB is 9, compute the area of $ABCD$.

Proposed by: Eric Neyman

Answer: 198

Let angles A, B, C, D have measures $90-3x, 90-x, 90+x, 90+3x$. Observe that angles B and C add up to 180 degrees, so $ABCD$ is a trapezoid with legs AB and CD. Let T and U be the feet of the altitudes to AB from C and D, respectively. Let $BT = y$ and $AU = z$. Then $\triangle BTC \sim \triangle CTA$, so $\frac{BT}{CT} = \frac{CT}{AT}$. We can write this as $\frac{y}{9} = \frac{9}{14+z}$, i.e. $y(14+z) = 81$. We also have $y = 9\tan x$ and $z = 9\tan 3x$. Writing $\tan 3x = \frac{3\tan x - \tan^3 x}{1-3\tan^2 x}$ (which can be verified easily using the tangent of sum formula), we find that $9\tan^4 x + 42\tan^3 x - 54\tan^2 x - 14\tan x + 9 = 0$. This seems daunting, but the rational root theorem lets us find the solution $\tan = \frac{1}{3}$. We know that $0 < x < 30°$, so $0 < \tan x < \frac{1}{\sqrt{3}}$. There are no other solutions to the quartic in this range. (Factoring out $3\tan x - 1$, we get $3\tan^3 x + 15\tan^2 x - 13\tan x - 9 = 0$. We find that this polynomial is positive at -1, negative at 0 and positive at 1, which means its remaining roots are less than -1; between -1 and 0; and greater than 1.)

Now that we have $\tan x = \frac{1}{3}$, we immediately get $y = 3$ and $z = 13$, so the area of the trapezoid is $9 \cdot \frac{14+30}{2} = \boxed{198}$.

13. Let $e_1, e_2, \ldots e_{2019}$ be independently chosen from the set $\{0, 1, \ldots, 20\}$ uniformly at random. Let $\omega = e^{\frac{2\pi}{i}2019}$. Determine the expected value of $|e_1\omega + e_2\omega^2 + \ldots + e_{2019}\omega^{2019}|^2$

Proposed by: Frank Lu

Answer: $\boxed{74030}$

First, we know that $\omega + \omega^2 + \ldots + \omega^{2019} = 0$, since $\omega \neq 1$ and ω is a root of $x^{2019} - 1 = 0$, which means that ω is a root of $1 + x + x^2 + \ldots + x^{2018}$. But then $\omega + \omega^2 + \ldots + \omega^{2019} = 0$. Thus, given a choice of e_i, subtracting 10 from each value of e_i yields the same value for the quantity $|e_1\omega + e_2\omega^2 + \ldots + e_{2019}\omega^{2019}|^2$ (and in fact gives the same complex number). We may thus shift the range of the e_i to be in $\{-10, -9, \ldots, 9, 10\}$. Now, suppose we are given the first n values of e_i, and we get the complex number z_n. We consider the expected value of $|z_n + e_{n+1}\omega^{n+1}|^2$. We first replace this with just any complex numbers $a+bi, c+di$ for generality. Then, the value of $|a+bi+e_{n+1}(c+di)|^2$ is $a^2 + b^2 + e_{n+1}^2c^2 + e_{n+1}^2d^2 + 2*a*c*e_{n+1} + 2*b*d*e_{n+1}$. Given our possible values for e_{n+1}, this yields us an average of $a^2 + b^2 + \frac{2}{21} * (0 + 1^2 + \ldots + 10^2)(c^2 + d^2)$. But we can calculate the coefficient as $\frac{2}{21} * \frac{10*11*21}{6} = \frac{110}{3}$. Hence, the expected increase in our absolute value due to any term is $\frac{110}{3}$. We thus see that starting with the value of 0 (starting with no terms at all), we get that our final answer is $\frac{110}{3} * 2019 = 74030$.

14. Consider a grid of black and white squares with 3 rows and n columns. If there is a non-empty sequence of white squares $s_1, ..., s_m$ such that s_1 is in the top row and s_m is in the bottom row and consecutive squares in the sequence share an edge, then we say that the grid percolates. Let T_n be the number of grids which do not percolate. There exists constants a, b such that $\frac{T_n}{ab^n} \to 1$ as $n \to \infty$. Then b is expressible as $(x + \sqrt{y})/z$ for squarefree y and coprime x, z. Find $x + y + z$.

Proposed by: Alan Yan

Answer: $\boxed{50}$

Let T_n be the number of non-percolating 3 by n grids, and let S_n be the number of nonpercolating 3 by n grids such that there is a sequence of white squares from the leftmost middle square (which must be white in S_n) to either the top or the bottom. Then we have that

$$S_n = 2(T_{n-1} - S_{n-1}) + 2S_{n-1} = 2T_{n-1}$$

by adding a column to the left, and

$$T_n = 7(T_{n-1} - S_{n-1}) + 6S_{n-1} = 7T_{n-1} - S_{n-1} = 7T_{n-1} - 2T_{n-2}$$

Thus we have that $b^2 = 7b - 2$, giving us $\frac{7+\sqrt{41}}{2}$.

15. Determine the number of functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ so that $\forall x \in \mathbb{Z}^+, f(f(x)) = f(x+1)$, and $\max(f(2), \ldots, f(14)) \leq f(1) - 2 = 12$.

Proposed by: Frank Lu

Answer: $\boxed{258}$

We replace 14 with k, or $f(1)$. We have that $f(k) = f(2)$, so it follows from above that $f(x) = f(x+k-2) \forall x \geq 2$. Now, suppose f was eventually periodic with period n as well. Let r be the remainder upon dividing $k-2$ by n, q the quotient. Then, $f(x+r) = f(x+q*n+r) =$

$f(x+(k-2)) = f(x) \forall x$ large enough. Hence, it follows that if r is the smallest value so $f(x) = f(x+r)$ eventually, it follows that r divides $k-2$. Combining the fact that $f(2) = f(k)$ yields that this eventual periodicity begins at $x = 2$. Let n be the smallest period of f. Now, it follows that f is determined by the values $f(1), f(2), \ldots, f(n), f(n+1)$. Suppose that $f(f(2))$ doesn't leave a remainder of 3 upon division by n. Then, it follows that $f(3) = f(x)$ for some x so $x-3$ isn't divisible by n. But running our argument above yields that n isn't our smallest period, contradiction. We can repeat this for other values. Thus, we need that $f(i)-(i+1)$ is divisible by i $\forall i \geq 2$. Now, we know that our period has to divide 12, suppose it is n. It follows that with period n, we have $(12/n)$ choices for each of $f(2), f(3), \ldots, f(n+1)$, (by the number of elements that are $i+1$ modulo $12/n$) which means we have $(12/n)^n$ possibilities. Summing this yields $12+6^2+4^3+3^4+2^6+1^{12} = 12+36+64+81+64+1 = 258$.