P U M a C

# Algebra A

1. Let $f(x) = \frac{x+a}{x+b}$ satisfy $f(f(f(x))) = x$ for real numbers $a$, $b$. If the maximum value of $a$ is $\frac{p}{q}$, where $p, q$ are relatively prime integers, what is $|p| + |q|$?

2. Let $C$ denote the curve $y^2 = \frac{x(x+1)(2x+1)}{6}$. The points $(\frac{1}{2}, a), (b, c)$, and $(24, d)$ lie on $C$ and are collinear, and $ad < 0$. Given that $b, c$ are rational numbers, find $100b^2 + c^2$.

3. Let $\{x\} = x - \lfloor x \rfloor$. Consider a function $f$ from the set $\{1, 2, \ldots, 2020\}$ to the half-open interval $[0, 1)$. Suppose that for all $x, y$, there exists a $z$ so that $\{f(x) + f(y)\} = f(z)$. We say that a pair of integers $m, n$ is valid if $1 \leq m, n \leq 2020$ and there exists a function $f$ satisfying the above so $f(1) = \frac{m}{n}$. Determine the sum over all valid pairs $m, n$ of $\frac{m}{n}$.

4. Let $P$ be a 10-degree monic polynomial with roots $r_1, r_2, \ldots, r_{10} \neq 0$ and let $Q$ be a 45-degree monic polynomial with roots $\frac{1}{r_i} + \frac{1}{r_j} - \frac{1}{r_i r_j}$ where $i < j$ and $i, j \in \{1, \ldots, 10\}$. If $P(0) = Q(1) = 2$, then $\log_2(|P(1)|)$ can be written as $\frac{a}{b}$ for relatively prime integers $a, b$. Find $a + b$.

5. Suppose we have a sequence $a_1, a_2, \ldots$ of positive real numbers so that for each positive integer $n$, we have that $\sum_{k=1}^{n} a_k a_{\lfloor \sqrt{k} \rfloor} = n^2$. Determine the first value of $k$ so $a_k > 100$.

6. Given integer $n$, let $W_n$ be the set of complex numbers of the form $re^{2qi\pi}$, where $q$ is a rational number so that $qn \in \mathbb{Z}$ and $r$ is a real number. Suppose that $p$ is a polynomial of degree $\geq 2$ such that there exists a non-constant function $f : W_n \to \mathbb{C}$ so that $p(f(x))p(f(y)) = f(xy)$ for all $x, y \in W_n$. If $p$ is the unique monic polynomial of lowest degree for which such an $f$ exists for $n = 65$, find $p(10)$.

7. Suppose that $p$ is the unique monic polynomial of minimal degree such that its coefficients are rational numbers and one of its roots is $\sin \frac{2\pi}{7} + \cos \frac{4\pi}{7}$. If $p(1) = \frac{a}{b}$, where $a, b$ are relatively prime integers, find $|a + b|$.

8. Let $a_n$ be the number of unordered sets of three distinct bijections $f, g, h : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ such that the composition of any two of the bijections equals the third. What is the largest value in the sequence $a_1, a_2, ...$ which is less than 2021?