Algebra B Solutions

1. The function $f(x) = x^2 + (2a+3)x + (a^2+1)$ only has real zeroes. Suppose the smallest possible value of a can be written in the form $\frac{p}{q}$, where p, q are relatively prime integers. Find $|p| + |q|$.

   Proposed by: Frank Lu

   Answer: $\boxed{17}$

   Notice that for $f(x)$ to have real zeroes, the discriminant needs to be nonnegative. This requires, in turn, that $(2a+3)^2 - 4(a^2+1) = 4a^2 + 12a + 9 - 4a^2 - 4 = 12a + 5$. But this requires, that $a \geq \frac{-5}{12}$. Our answer is hence $\boxed{17}$.

2. Princeton has an endowment of 5 million dollars and wants to invest it into improving campus life. The university has three options: it can either invest in improving the dorms, campus parties or dining hall food quality. If they invest a million dollars in the dorms, the students will spend an additional $5a$ hours per week studying. If the university invests b million dollars in better food, the students will spend an additional $3b$ hours per week studying. Finally, if the c million dollars are invested in parties, students will be more relaxed and spend $11c - c^2$ more hours per week studying. The university wants to invest its 5 million dollars so that the students get as many additional hours of studying as possible. What is the maximal amount that students get to study?

   Proposed by: Aleksa Milojevic

   Answer: $\boxed{34}$

   Note: On the original algebra test, we had forgotten the "per week" on the last quantity of hours.

   We have to maximize $5a + 3b + 11c - c^2$ when $a + b + c = 5$. It is clear that $b = 0$ as it is always better to invest in a than in b. We know $5a + 11c - c^2 = 5(a+c) + (6c - c^2) = 5 \cdot 5 + 9 - (c-3)^2 = 34 - (c-3)^2$. The optimal choice is $a = 2, b = 0, c = 3$; our answer is $\boxed{34}$.

3. Let $f(x) = \frac{x+a}{x+b}$ satisfy $f(f(f(x))) = x$ for real numbers a, b. If the maximum value of a is $\frac{p}{q}$, where p, q are relatively prime integers, what is $|p| + |q|$?

   Proposed by: Henry Erdman

   Answer: $\boxed{7}$

   Substituting in $f(x)$ for x in $f(x)$ twice yields that $f(f(f(x))) = \frac{(1+2a+ab)x+(a+a^2+ab+ab^2)}{(1+a+b+b^2)x+(a+2ab+b^3)}$. We note that the coefficient of x in the denominator must be zero, and thus we have that $a = -b^2 - b - 1$. This parabola opens down and has its vertex at $b = -\frac{1}{2}$, giving an upper limit on a of $-\frac{3}{4}$. We now need to verify that $(a, b) = (-\frac{3}{4}, -\frac{1}{2})$ satisfies the rest of the problem. We have $1 + 2a + ab = 1 - \frac{3}{2} + \frac{3}{8} = -\frac{1}{8}$ as the coefficient of x in the numerator, $-\frac{3}{4} + \frac{9}{16} + \frac{3}{8} - \frac{3}{16} = 0$ as the constant in the numerator, and $-\frac{3}{4} + \frac{3}{4} - \frac{1}{8} = -\frac{1}{8}$ as the constant in the denominator. Thus, we do indeed have a solution, and it is the greatest possible value of a. So, our answer is $|-3| + |4| = \boxed{7}$.

4. Let C denote the curve $y^2 = \frac{x(x+1)(2x+1)}{6}$. The points $(\frac{1}{2}, a), (b, c)$, and $(24, d)$ lie on C and are collinear, and $ad < 0$. Given that b, c are rational numbers, find $100b^2 + c^2$.

   Proposed by: Sunay Joshi

   Answer: $\boxed{101}$

By plugging $x = \frac{1}{2}$ into the equation for C, we find $a = \mp\frac{1}{2}$. Similarly, $d = \pm 70$. By geometric intuition (?), there are only two possible pairs (a, d), namely $(a, d) = (-\frac{1}{2}, 70)$ or $(\frac{1}{2}, -70)$.

Suppose $(a, d) = (-\frac{1}{2}, 70)$. Then the equation of the line through $(\frac{1}{2}, -\frac{1}{2})$ and $(24, 70)$ is $y = 3x-2$. Plugging this into the equation for C, we find $(3x-2)^2 = \frac{x(x+1)(2x+1)}{6}$. Simplifying, we find $2x^3 - 51x^2 + \ldots = 0$.

At this point, instead of solving this equation explicitly, we use a trick. Since $(\frac{1}{2}, -\frac{1}{2})$ and $(24, -70)$ lie on this line, $x = \frac{1}{2}$ and $x = 24$ are roots of this cubic. Thus, the remaining root $x = b$ must satisfy Vieta's Formula for the sum of roots! We get $b + \frac{1}{2} + 24 = \frac{51}{2}$, thus $b = 1$. Plugging this into the equation of our line, we find $c = 1$, hence $(b, c) = (1, 1)$.

By the symmetry of C across the x axis, the other case yields $(b, c) = (1, -1)$. In either case, we find an answer of $100 \cdot 1^2 + 1^2 = \boxed{101}$.

5. Let $\{x\} = x - \lfloor x \rfloor$. Consider a function f from the set $\{1, 2, \ldots, 2020\}$ to the half-open interval $[0, 1)$. Suppose that for all x, y, there exists a z so that $\{f(x) + f(y)\} = f(z)$. We say that a pair of integers m, n is valid if $1 \leq m, n, \leq 2020$ and there exists a function f satisfying the above so $f(1) = \frac{m}{n}$. Determine the sum over all valid pairs m, n of $\frac{m}{n}$

   Proposed by: Frank Lu

   Answer: $\boxed{1019595}$

   We will consider the set of all possible images for f, as this is the only restriction we are given on our function.

   First, suppose that $f(x)$ was irrational for some value of x. Then, it follows that $\{n * f(x)\}$ is in the image of f for all $n \in \mathbb{N}$. But this is impossible since our domain has only finitely many elements. Thus, it follows that our function can only be rational-valued. By repeating this argument, we also know that the denominator of $f(x)$ must be at most 2020.

   We now claim that all such values are valid for $f(1)$. To see this, let $f(x) = \{xf(1)\}$. The fact that our condition is satisfied is clear. We thus find $\sum_{i=1}^{2020} \sum_{j=0}^{i-1} \frac{j}{i} = \sum_{i=1}^{2020} \frac{i-1}{2} = 2019 * 505 = \boxed{1019595}$ is our answer.

6. Let P be a 10-degree monic polynomial with roots $r_1, r_2, \ldots, r_{10} \neq 0$ and let Q be a 45-degree monic polynomial with roots $\frac{1}{r_i} + \frac{1}{r_j} - \frac{1}{r_i r_j}$ where $i < j$ and $i, j \in \{1, \ldots, 10\}$. If $P(0) = Q(1) = 2$, then $\log_2(|P(1)|)$ can be written as $\frac{a}{b}$ for relatively prime integers a, b. Find $a + b$.

   Proposed by: Matthew Kendall

   Answer: $\boxed{19}$

   We can factor Q as a product of its roots:

$$Q(x) = \prod_{i<j} \left(x - \frac{1}{r_i} - \frac{1}{r_j} + \frac{1}{r_i r_j} \right).$$

   Then we see

$$Q(1) = \prod_{i<j} \left(1 - \frac{1}{r_i} - \frac{1}{r_j} + \frac{1}{r_i r_j} \right) = \prod_{i<j} \frac{1}{r_i r_j}(1 - r_i)(1 - r_j) = \frac{1}{(r_1 r_2 \cdots r_{10})^9} P(1)^9.$$

   Hence $\frac{1}{2^9}|P(1)|^9 = 2$, so $|P(1)| = 2^{\frac{10}{9}}$, giving an answer of $\boxed{19}$.

7. Suppose we have a sequence $a_1, a_2, \ldots$ of positive real numbers so that for each positive integer n, we have that $\sum_{k=1}^{n} a_k a_{\lfloor \sqrt{k} \rfloor} = n^2$. Determine the first value of k so $a_k > 100$.

   Proposed by: Frank Lu

   Answer: $\boxed{1018}$

   Note: On the original algebra test, we had forgotten to include the phrase "positive real numbers."

   Notice that this relation becomes the equation that $a_n = \frac{2n-1}{a_{\lfloor \sqrt{n} \rfloor}}$, by subtracting this for n and $n-1$. From here, to figure out when this is larger than 100, we need to make some deductions about the rough behavior of this sequence. Notice here that, trying smaller values, we have that $a_2 = 3, a_3 = 5, a_4 = 7/3, a_5 = 3, a_6 = 11/3$.

   First, notice that $a_n = \frac{2n-1}{2\lfloor \sqrt{n} \rfloor - 1} a_{\lfloor \sqrt{\lfloor \sqrt{n} \rfloor} \rfloor} > \sqrt{n} a_{\lfloor \sqrt{\lfloor \sqrt{n} \rfloor} \rfloor}$. Observe then that for $n = 1295 = 6^4 - 1$, notice that $a_{1295} > 35 a_5 = 105$, so hence our maximal value is going to be at most 1295. In particular, we see that $a_{\lfloor \sqrt{\lfloor \sqrt{n} \rfloor} \rfloor}$ for our maximal value is either going to be a_1, a_2, a_3, a_4, or a_5. But notice however that $\lfloor \sqrt{\lfloor \sqrt{n} \rfloor} \rfloor = 5$; if it is 4, notice that this is at most $\frac{7}{3} \frac{2n-1}{2\lfloor \sqrt{n} \rfloor - 1} < \frac{7}{3} \frac{1250}{31} = \frac{8750}{93} < 100$. And furthermore, if it is less than 4, we see that we can bound this more crudely by $5 \frac{2n-1}{2\lfloor \sqrt{n} \rfloor - 1} < 5 \frac{2(\lfloor \sqrt{n} \rfloor + 1)^2 - 1}{2\lfloor \sqrt{n} \rfloor - 1} = 5 \frac{2\lfloor \sqrt{n} \rfloor^2 + 4\lfloor \sqrt{n} \rfloor + 1}{2\lfloor \sqrt{n} \rfloor - 1} = 5(\lfloor \sqrt{n} \rfloor + 5/2 + \frac{7/2}{2\lfloor \sqrt{n} \rfloor - 1})$. On the one hand, we see that if $\lfloor \sqrt{n} \rfloor \leq 3$, this is at most $5(3 + 5/2 + 7/2) < 45$. On the other hand, $\lfloor \sqrt{n} \rfloor \geq 4$, so this is at most $5(15 + 5/2 + 1/2) < 90 < 100$.

   In particular, we require then that for our minimal value for n, we have that $a_n = \frac{6n-3}{2\lfloor \sqrt{n} \rfloor - 1}$. On one hand, again we can use our bounds above to see that this is bounded above by $3(\lfloor \sqrt{n} \rfloor + 5/2 + \frac{7/2}{2\lfloor \sqrt{n} \rfloor - 1})$; we therefore need to have that $\lfloor \sqrt{n} \rfloor + 5/2 + \frac{7/2}{2\lfloor \sqrt{n} \rfloor - 1} > 33$. But with $\sqrt{n} \geq 25$ in this particular subcase, this means that we have that $\lfloor \sqrt{n} \rfloor + 3 > 33$, or that $\lfloor \sqrt{n} \rfloor > 30$. We start with this being 31; we then get that $a_n = \frac{6n-3}{61}$. To be larger than 100, this requires that $6n > 6103$, or that $n \geq \boxed{1018}$.

8. Given integer n, let W_n be the set of complex numbers of the form $re^{2qi\pi}$, where q is a rational number so that $qn \in \mathbb{Z}$ and r is a real number. Suppose that p is a polynomial of degree ≥ 2 such that there exists a non-constant function $f : W_n \to \mathbb{C}$ so that $p(f(x))p(f(y)) = f(xy)$ for all $x, y \in W_n$. If p is the unique monic polynomial of lowest degree for which such an f exists for $n = 65$, find $p(10)$.

   Proposed by: Frank Lu

   Answer: $\boxed{100009}$

   Note: On the original algebra test, we had forgotten the phrase " r is a real number."

   Fix $f(1)$ and $p(x)$.

   First, note that plugging in $x = y = 1$ yields that $p(f(1))^2 = f(1)$, and $y = 1$ yields that $p(f(x))p(f(1)) = f(x)$.

   Hence, we see that the image of f is a root of the polynomial $p(u)p(f(1)) - u = 0$, which in particular means that f has a finite image. Furthermore, we thus see that $p(f(x))p(f(y))p(f(1))^2 = f(xy)p(f(1))^2$, which means that, in fact, that $f(x)f(y) = f(1)f(xy) \forall x, y \in W_n$. If $f(1)$ is zero, then it follows that $\forall x \in W_n$ that $f(x) = 0$, so we consider when $f(1) \neq 0$. Then, we see that, letting $g(x) = f(x)/f(1)$ that $g(x)g(y) = g(xy) \forall x, y \in W_n$.

   Since the image of g is finite, if there exists a value of x so that $|g(x)| \neq 1, |g(x)| \neq 0$, then $g(x), g(x^2), \ldots$ are all distinct, contradiction. Furthermore, $g(x) = 0$ for some $x \neq 0$ means that $g(x)g(y/x) = 0 = g(y) \forall y \in W_n$, so we take that we want $|g(x)| = 1$ for all $x \in W_n$. By a similar logic, we see that $g(x)$ must be a root of unity, as again we will run into the issue where the image of g is infinite.

We see that if p is a prime not dividing n, then $g(x)$ can't ever be a pth root of unity, since otherwise we could take the pth root of x to get another root (raising all of the roots to the pth power yields a permutation of the roots). Thus, we see that the minimal possible value for the degree of our polynomial is 5, which would then require it to have the 5th roots of unity as roots.

Thus, we see that $p(u)p(f(1)) - u = au^5 - a$ for some complex number a, meaning that $p(u) = \frac{a}{p(f(1))}u^5 + u - \frac{a}{p(f(1))}$, which we can just write as $p(u) = u + c(u^5 - 1)$ for some complex constant c. By monic, we see that $p(x) = x^5 + x - 1$ yields that $p(10) = \boxed{100009}$.