Geometry B Solutions

1. You are walking along a road of constant width with sidewalks on each side. You can only walk on the sidewalks or cross the road perpendicular to the sidewalk. Coming up on a turn, you realize that you are on the "outside" of the turn; i.e., you are taking the longer way around the turn. The turn is a circular arc. Assuming that your destination is on the same side of the road as you are currently, let θ be the smallest turn angle, in radians, that would justify crossing the road and then crossing back after the turn to take the shorter total path to your destination. What is $\lfloor 100 \times \theta \rfloor$?

   Proposed by: Henry Erdman

   Answer: $\boxed{200}$

   Let the radius of the turn be r and the width of the road w. Then, for a turn of angle θ, the outside path has length $(r+w)\theta$. The inside path has length $2w + r\theta$. These are equal when $\theta = 2$, so our answer is 200.

2. Seven students in Princeton Juggling Club are searching for a room to meet in. However, they must stay at least 6 feet apart from each other, and due to midterms, the only open rooms they can find are circular. In feet, what is the smallest diameter of any circle which can contain seven points, all of which are at least 6 feet apart from each other?

   Proposed by: Daniel Carter

   Answer: $\boxed{12}$

   The optimal arrangement is one person in the middle with six surrounding them in a regular hexagon, giving a diameter of 12 feet.

3. Let γ_1 and γ_2 be circles centered at O and P respectively, and externally tangent to each other at point Q. Draw point D on γ_1 and point E on γ_2 such that line DE is tangent to both circles. If the length $OQ = 1$ and the area of the quadrilateral $ODEP$ is 520, then what is the value of length PQ?

   Proposed by: Ollie Thakar

   Answer: $\boxed{64}$

   Let r be the radius OQ of γ_1 and s the radius PQ of γ_2.

   It is a well-known theorem that angle $\angle EQD$ is right, and the length of the hypotenuse ED is $2\sqrt{rs}$.

   Call $a = EQ$ and $b = DQ$. Call x the measure of angle $\angle DQO$. Then, $\angle EQP$ has measure $90 - x$. Furthermore, triangles DOQ and EPQ are isosceles, so $a = 2s\sin x$ and $b = 2s\cos x$.

   Since triangle EQD is right, we have $a^2 + b^2 = ED^2$, which gives us $\sin^2 x = \frac{r}{r+s}$ and $\cos^2 x = \frac{s}{r+s}$.

   The area A of quadrilateral $ODEP$ is given by the sum of the areas of triangles DOQ, EPQ, and EDQ, so:

   $$A = \frac{1}{2}(s\cos x)(2s\sin x) + \frac{1}{2}(r\sin x)(2r\cos x) + \frac{1}{2}4rs\sin x\cos x = (r+s)^2\sin x\cos x = (r+s)\sqrt{rs}.$$

   We are given that $r = 1$ and then that $(1+s)\sqrt{s} = 520$, which can be solved as $s = 64$ by inspection.

4. Hexagon $ABCDEF$ has an inscribed circle Ω that is tangent to each of its sides. If $AB = 12$, $\angle FAB = 120°$, and $\angle ABC = 150°$, and if the radius of Ω can be written as $m + \sqrt{n}$ for positive integers m, n, find $m + n$.

   Proposed by: Sunay Joshi

   Answer: $\boxed{36}$

   Let r denote the radius of Ω, let O denote the center of Ω, and let Ω touch side AB at point X. Then OX is the altitude from O in $\triangle AOB$. Note that $\angle OAB = \frac{1}{2}\angle FAB = 60°$ and $\angle OBA = \frac{1}{2}\angle ABC = 75°$. Thus by right angle trigonometry, $AX = \frac{r}{\tan 60°} = \frac{\sqrt{3}}{3}r$ and $BX = \frac{r}{\tan 75°} = (2-\sqrt{3})r$. As $AB = AX + BX = 12$, we have $(\frac{\sqrt{3}}{3} + 2 - \sqrt{3})r = 12 \to r = 9 + \sqrt{27}$, thus our answer is $m + n = 36$.

5. Let $ABCD$ be a cyclic quadrilateral with circumcenter O and radius 10. Let sides AB, BC, CD, and DA have midpoints M, N, P, and Q, respectively. If $MP = NQ$ and $OM + OP = 16$, then what is the area of triangle $\triangle OAB$?

   Proposed by: Ollie Thakar

   Answer: $\boxed{78}$

   Note: The configuration provided in this problem turned out to be impossible, since we arrive at the condition $OM^2 + OP^2 = 100$, *which cannot hold with the given condition that* $OM + OP = 16$. *As such, this problem was thrown out during the competition.*

   The condition that $MP = NQ$ is equivalent to the condition that $AC \perp BD$. (This can be seen because the quadrilateral $MNPQ$ is a parallelogram whose sides are parallel to the diagonals AC and BD. The condition $MP = NQ$ implies that the parallelogram has equal diagonals, so is a rectangle.) Let r be the circumradius of $ABCD$. By two well-known properties of cyclic orthodiagonal quadrilaterals, we get: $r^2 = AM^2 + CP^2$, $OP = AM$, and $OM = CP$.

   Then, $\text{Area}(\triangle OAB) = \frac{1}{2}OM \cdot AB = OM \cdot OP$, and $r^2 = OP^2 + OM^2$.

   Thus,

$$\text{Area}(\triangle OAB) = OM \cdot OP = \frac{1}{2}\left((OP+OM)^2 - (OP^2+OM^2)\right) = \frac{1}{2}\left((OP+OM)^2 - r^2\right) = \frac{1}{2}(16^2 - 10^2) = 78.$$

6. Let C be a circle centered at point O, and let P be a point in the interior of C. Let Q be a point on the circumference of C such that $PQ \perp OP$, and let D be the circle with diameter PQ. Consider a circle tangent to C whose circumference passes through point P. Let the curve Γ be the locus of the centers of all such circles. If the area enclosed by Γ is $1/100$ the area of C, then what is the ratio of the area of C to the area of D?

   Proposed by: Ollie Thakar

   Answer: $\boxed{2500}$

   Let r be the radius of C, and let the length $OP = x$.

   First, we prove that Γ is an ellipse with foci at O and P. Let X be a point on Γ. Then, draw a circle E centered at X passing through point P, tangent to C. Since C and E are tangent circles, then O, X, and C are collinear. But $XC = XP$, so $r = OC = OX + XC = OX + XP$, so $OX + XP$ is a constant for all X on the curve Γ, which is the definition of an ellipse.

   The area of Γ is equal to π times the semi-major axis times the semi-minor axis, or, after an application of the Pythagorean theorem: $\pi \cdot \frac{r}{2} \cdot \frac{1}{2}\sqrt{r^2 - x^2}$.

   Also by the Pythagorean Theorem, $QP^2 = r^2 - x^2$, so that means the area of Γ is $\frac{\pi}{4}rQP$.

By the condition that the area of C is 100 times that of Γ, then we get that $\pi r^2 = 100\frac{\pi}{4}rQP$, from which we conclude that $\frac{r}{QP} = \frac{100}{4} = 25$, but the ratio of the area of C to the area of D is precisely the square of the ratio $\frac{2r}{QP}$, which is $(2 \cdot 25)^2 = 2500$.

Note: We initially had the answer of 625, *but this is incorrect on account of QP being the diameter and not the radius of the circle. We apologize for the confusion this would have caused.*

7. Triangle ABC is so that $AB = 15, BC = 22$, and $AC = 20$. Let D, E, F lie on BC, AC, and AB, respectively, so AD, BE, CF all contain a point K. Let L be the second intersection of the circumcircles of BFK and CEK. Suppose that $\frac{AK}{KD} = \frac{11}{7}$, and $BD = 6$. If $KL^2 = \frac{a}{b}$, where a, b are relatively prime integers, find $a + b$.

   Proposed by: Frank Lu

   Answer: $\boxed{497}$

   First, by Menalaus's theorem, we can compute that $\frac{AK}{KD}\frac{DC}{CB}\frac{BF}{FA} = 1$, which in turn implies that $\frac{BF}{FA} = \frac{7}{11}\frac{22}{16} = \frac{7}{8}$. Therefore, by Ceva's theorem, it follows that $\frac{AE}{EC} = \frac{AF}{FB}\frac{BD}{DC} = \frac{8}{7}\frac{6}{16} = \frac{3}{7}$. From here, we see that $AF = 8, AE = 6$. In particular, notice that by power of point, since $AE \cdot AC = 120 = AB \cdot AF$, it follows that A lies on the radical axes of these circles; in particular, notice that A, K, L are collinear.

   Now, notice that the length of AD, by Stewart's theorem, is so that $BD \cdot DC \cdot BC + AD^2 \cdot BC = AC^2 \cdot BD + AB^2 \cdot CD$. Plugging in the values we computed, it follows that $6 \cdot 16 \cdot 22 + AD^2 \cdot 22 = 20^2 \cdot 6 + 15^2 \cdot 16 = 3600 + 2400 = 6000$. In particular, it follows that $AD^2 = \frac{6000-96\cdot 22}{22} = \frac{3888}{22} = \frac{1944}{11}$, or that $AD = 18\sqrt{\frac{6}{11}}$. In particular, this means that $AK = \sqrt{66}$. Therefore, computing the power of A again, we see that $AK \cdot AL = 120$ too, meaning that it follows that $AL = \frac{120}{\sqrt{66}} = \frac{20\sqrt{66}}{11}$. Hence, it follows that $KL = \frac{9\sqrt{66}}{11}$, and so that $KL^2 = \frac{486}{11} = 497$.

8. Triangle ABC has side lengths 13, 14, and 15. Let E be the ellipse that encloses the smallest area which passes through A, B, and C. The area of E is of the form $\frac{a\sqrt{b}\pi}{c}$, where a and c are coprime and b has no square factors. Find $a + b + c$.

   Proposed by: Daniel Carter

   Answer: $\boxed{118}$

   Let T be an affine transformation that sends an equilateral triangle with side length 1 to triangle ABC. Affine transformations preserve the ratios of areas, so the smallest such ellipse for the equilateral triangle will be sent to E by T. It is clear by inspection that the smallest area ellipse for the equilateral triangle is its circumcircle. The circumcircle of an equilateral triangle has area $\frac{4\sqrt{3}\pi}{9}$ times the area of the triangle, and the area of ABC is 84 (found via Heron's formula), so the area of the E is $\frac{4\sqrt{3}\pi}{9} \cdot 84 = \frac{112\sqrt{3}\pi}{3}$. Thus the answer is $112 + 3 + 3 = 118$.