Individual Finals B

1. Let $a_1, \ldots, a_{2020}$ be a sequence of real numbers such that $a_1 = 2^{-2019}$, and $a_{n-1}^2 a_n = a_n - a_{n-1}$. Prove that $a_{2020} < \dfrac{1}{2^{2019}-1}$.

   Proof. We will prove by induction that $0 < a_i \leq 2^{-2020+i}$ for $i = 1, \ldots, 2020$. The base follows from the definition of a_1.

   Suppose the statement holds for i. Then $a_{i+1} = \dfrac{a_i}{1-a_i^2}$, from the recurrence equation. By the inductive hypothesis, first we see that $a_{i+1} > 0$. Furthermore, $\dfrac{a_i}{1-a_i^2} \leq 2^{-2020+i}\dfrac{1}{1-a_i^2}$. It is enough to prove that $\dfrac{1}{1-a_i^2} < 2$ which follows from $a_i < 2^{-2020+i}$.

   From the recurrence equation, we get that $a_{n-1} = \dfrac{1}{a_{n-1}} - \dfrac{1}{a_n}$. Then $\dfrac{1}{a_1} - \dfrac{1}{a_{2020}} = \sum_{i=1}^{2019} a_i \leq \sum_{i=1}^{2019} 2^{-2020+i} < 1$. Then $2^{2019} - \dfrac{1}{a_{2020}} < 1$, from which it follows that $a_{2020} < \dfrac{1}{2^{2019}-1}$.

   Remark: This technique can be repeated to get an even better estimate by using a similar estimate for $a_i < \dfrac{1}{2^{2019}-1}$, yielding $\sum_{i=1}^{2019} a_i < \dfrac{2019}{2^{2019}-1}$.

   Proposed by Aleksa Milojević and Igor Medvedev.

2. Helen has a wooden rectangle of unknown dimensions, a straightedge, and a pencil (no compass). Is it possible for her to construct a line segment on the rectangle connecting the midpoints of two opposite sides, where she cannot draw any lines or points outside the rectangle?

   Note: Helen is allowed to draw lines between two points she has already marked, and mark the intersection of any two lines she has already drawn, if the intersection lies on the rectangle. Further, Helen is allowed to mark arbitrary points either on the rectangle or on a segment she has previously drawn. Assume that only the four vertices of the rectangle have been marked prior to the beginning of this process.

   Solution: We will show that we can construct the midpoint of any edge of the rectangle. First, we draw the diagonals, and mark their intersection E.

   Then, we pick an arbitrary point X on AE, not being A or E. Then, we intersect DX with AB to get Y and EY with BX to get Z. The claim is that the pencil of lines $(AB, AE; AZ, AD)$ are harmonic. To see why, it suffices to let T be the intersection of AZ and BE. From Ceva's theorem in triangle ABE we have that $\frac{\overrightarrow{BT}}{\overrightarrow{TE}} \cdot \frac{\overrightarrow{EX}}{\overrightarrow{XA}} \cdot \frac{\overrightarrow{AY}}{\overrightarrow{YB}} = 1$. Similarly, Menelaus' theorem in triangle ABE gives $\frac{\overrightarrow{BD}}{\overrightarrow{DE}} \cdot \frac{\overrightarrow{EX}}{\overrightarrow{XA}} \cdot \frac{\overrightarrow{AY}}{\overrightarrow{YB}} = -1$. These equations imply $\frac{\overrightarrow{BD}}{\overrightarrow{DE}} = -\frac{\overrightarrow{BT}}{\overrightarrow{TE}}$, which gives the above claim.

   This means that the line AZ passes through the midpoint of BC (because $AD||BC$, and projecting the harmonic pencil $(AB, AE; AD, AZ)$ onto BC gives respectively the B, C, the point at infinity and the midpoint of BC). Other midpoints can be constructed similarly.

   Proposed by Daniel Carter.

3. Let n be a positive integer, and let $\mathcal{F}$ be a family of subsets of $\{1, 2, \cdots, 2^n\}$ such that for any non-empty $A \in \mathcal{F}$ there exists $B \in \mathcal{F}$ so that $|A| = |B| + 1$ and $B \subset A$. Suppose that $\mathcal{F}$

contains all (2^n-1)-element subsets of $\{1,2,\cdots,2^n\}$. Determine the minimal possible value of $|\mathcal{F}|$.

Solution: The answer is $n\cdot 2^n+1$. First we will provide a construction for this answer, inductively. For $n=1$, we can obviously construct $\mathcal{F}=\{\varnothing,\{1\},\{2\}\}$, which has a cardinality of $3=1\cdot 2^1+1$. For larger n, we let $\mathcal{F}_1$ be the solution for $n-1$ and every set also contains the numbers $\{2^{n-1}+1,2^{n-1}+2,\cdots,2^n\}$ and let $\mathcal{F}_2$ be the family symmetrical to $\mathcal{F}_1$ in the sense that if we will replace every element x with 2^n+1-x. Now let

$$\mathcal{F}=\mathcal{F}_1\cup\mathcal{F}_2\cup$$
$$\big\{\varnothing,\{1\},\{1,2\},\cdots,\{1,\cdots,2^{n-1}-1\},\{2^{n-1}+1\},\{2^{n-1}+1,2^{n-1}+2\},\cdots,\{2^{n-1}+1,\cdots,2^n-1\}\big\}.$$

By the inductive hypothesis, $\mathcal{F}$ obviously satisfies all the required conditions. Also

$$|\mathcal{F}|=2\cdot|\mathcal{F}_1|+2^n-1=$$
$$2\cdot((n-1)\cdot 2^{n-1}+1)+2^n-1=(n-1)\cdot 2^n+2^n+1=n\cdot 2^n+1.$$

Now we will prove this number is minimal. Let $\mathcal{F}_m$ be a family that satisfies the problem condition, which has the minimal possible number of sets. Obviously this family will contain the empty set. We shall construct a rooted tree T in the following way: the vertices will represent elements of $\mathcal{F}_m$, and the parent of the vertex that corresponds to the set $A\in\mathcal{F}_m$ will be a vertex that corresponds to a $B\in\mathcal{F}_m$ so that $|A|=|B|+1$ and $B\subset A$ (there can be multiple such B, but only a single, arbitrary one will be chosen as the parent). As every vertex except the one that corresponds to the empty set have a parent, T is rooted in that vertex. Now, since we have assumed the minimality of $\mathcal{F}_m$, we can see that the only leaves in this tree are the vertices that correspond to the (2^n-1)-element sets. Let the *height* of a vertex be the vertex-wise distance from it to the nearest leaf, and deonte the height of the vertex corresponding to A as h_A. Let the *power* of a vertex denote the number of leaves in its subtree, and denote the power of the vertex corresponding to A as x_A. Now observe the following:

Lemma 1: $h_A\geq x_A$ for any $A\in\mathcal{F}_m$.

Notice that $h_A=2^n-|A|$, so there are exactly h_A numbers from $\{1,2,\cdots,2^n\}$ that are not in A. We notice that if if the vertex corresponding to C in the subtree of the vertex corresponding to A, then $A\subset C$. This means that the only leaves that can be in this subtree are the ones whose missing element of the corresponding set isn't in A. From this we derive the desired inequality.

Lemma 2: For any $A\in\mathcal{F}_m$ let's denote the number of vertices in the subtree of its corresponding vertex with p_A. Then we have

$$p_A\geq x_A\log_2 x_A+h_A-x_A+1.$$

We will prove this lemma by induction on the value of x_A+h_A. The base case of $x_A+h_A=2$ only holds when $x_A=1$ and $h_A=1$, meaning that A corresponds to a leaf, for which the lemma is obviously true. Now for the inductive step we lets denote the sons of set A by $B_1,B_2,\cdots,B_k$. Obviously, we have $h_{B_k}=h_A-1$ and $\sum_{i=1}^k x_{B_k}=x_A$. In this case we see

$$p_A=1+\sum_{i=1}^k p_{B_k}\geq\sum_{i=1}^k x_{B_k}\log_2 x_{B_k}+\sum_{i=1}^k h_{B_k}-\sum_{i=1}^k x_{B_k}+k+1\geq\sum_{i=1}^k x_{B_k}\log_2 x_{B_k}+k\cdot h_A-x_A+1.$$

Now since $f(x) = x \ln x$ is convex, by Jensen's inequality we obtain

$$p_A \geq k \cdot \frac{\sum_{i=1}^{k} x_{B_i}}{k} \log_2 \left(\frac{\sum_{i=1}^{k} x_{B_i}}{k} \right) + h_A - x_A + 1 + (k-1) h_A \geq x_A \log_2 \left(\frac{x_A}{k} \right) + (k-1) x_A + h_A - x_A + 1.$$

Since $x_A(\log_2 \frac{x_A}{k} + (k-1)) = x_A \log_2(x_A \frac{2^{k-1}}{k}) \geq x_A \log_2(x_A)$, because $2^{k-1} = (1+1)^{k-1} \geq 1 + k - 1 = k$ by Bernoulli's inequality (also provable by induction). With this we have proven the lemma.

Now applying *Lemma 2* to the root vertex we obtain that the amount of vertices is at least

$$p_\varnothing \geq 2^n \cdot n + 2^n - 2^n + 1 = n \cdot 2^n + 1.$$

Proposed by Pavle Martinović.