P U M ∴ C

Individual Finals B

1. Find all pairs of natural numbers (n, k) with the following property:

   Given a $k \times k$ array of cells, such that every cell contains one integer, there always exists a path from the left to the right edges such that the sum of the numbers on the path is a multiple of n.

   Note: A path from the left to the right edge is a sequence of cells of the array $a_1, a_2, \ldots, a_m$ so that a_1 is a cell of the leftmost column, a_m is the cell of the rightmost column, and a_i, a_{i+1} share an edge for all $i = 1, 2, \ldots, m-1$.

   Answer: The pair (n, k) satisfies the above property if and only if $n \leq k$.

   Solution: The proof consists of two parts. In case $n > k$, we will construct an array of cells which does not has the above property, while in case $n \leq k$ we will prove that the property always holds.

   If $n > k$, consider the following array: let all the columns but the rightmost one be filled with all zeroes and the rightmost with all ones. Then every path from the left edge to the right edge has the sum at least one, and at most k. As $n > k$, this means none of the possible sums is divisible by n, which completes the construction.

   In case $n \leq k$, let the sum of numbers in the row i be R_i. We will first prove that there is a there is a contiguous segment $R_i, ..., R_j$ such that the sum $R_i + \cdots + R_j$ is divisible by n. Consider the sums $S_1 = R_1, S_2 = R_1 + R_2, \ldots, S_k = R_1 + \cdots + S_k$. Then, either one of the S_i is divisible by n or all of them have non-zero residues modulo n. In the first case, the segment $R_1, \ldots, R_i$ is a contiguous segment satisfying the above property. In case all residues of S_i are non-zero, by Pigeonhole principle, there must be two sums which have the same residue, say $S_i \equiv S_j (\mod n)$. This means that $n | S_j - S_i = R_{i+1} + \cdots + R_j$, which provides the wanted contiguous segment.

   Now, it is easy to construct a path passing only the cells from the rows $i, \ldots, j$. It suffices to go column by column, passing the whole column before going onto the next one.

   Proposed by Pavle Martinović.

2. Prove that there is a positive integer M for which the following statement holds:

   For all prime numbers p, there is an integer n for which $\sqrt{p} \leq n \leq M\sqrt{p}$ and $p \mod n \leq \frac{n}{2020}$.

   Note: Here, $p \mod n$ denotes the unique integer $r \in \{0, 1, \ldots, n-1\}$ for which $n | p - r$. In other words, $p \mod n$ is the residue of p upon division by n.

   Solution: We will show that any $M > 4040^2$ satisfies the conditions of the problem.

   First, we will consider primes with $p > 4040^2$. We will show that any $M > 4040$ works here. Because $\frac{\sqrt{p}}{2020} \geq 2$, there is an integer q in the interval $[\frac{\sqrt{p}}{4040}, \frac{\sqrt{p}}{2020}]$. As q is at most $\frac{\sqrt{p}}{2020}$, it divides an integer s in the interval $[p - \frac{\sqrt{p}}{2020}, p-1]$, as the length of this interval is bigger than q. Pick $n = \frac{s}{q}$. Then, $n \geq \frac{p - \frac{\sqrt{p}}{2020}}{\frac{\sqrt{p}}{2020}} = 2020(\sqrt{p} - \frac{1}{2020}) \geq \sqrt{p}$. Similarly, $n \leq \frac{p}{q} \leq 4040\sqrt{p} < M\sqrt{p}$. Finally, $p \mod n \leq p - s \leq \frac{\sqrt{p}}{2020} \leq \frac{n}{2020}$.

   Now, having proven that any $M > 4040$ works in case $p \geq 4040^2$, we can consider the primes $p \leq 4040^2$ as well. For them, it suffices to choose $M > 4040^2$ as well, because one can just pick $n = p$, and it will satisfy the conditions of the problem: $p \mod n = 0$, $n = p \geq \sqrt{p}$ and $n = \sqrt{p}\sqrt{p} \leq 4040\sqrt{p} \leq M\sqrt{p}$.

   Therefore, any integer $M > 4040^2$ satisfies the conditions of this problem.

Proposed by Aleksa Milojević.

3. Let ABC be a triangle and let the points D, E be on the rays AB, AC such that $BCED$ is cyclic. Prove that the following two statements are equivalent:

   - There is a point X on the circumcircle of ABC such that BDX, CEX are tangent to each other.
   - $AB \cdot AD \leq 4R^2$, where R is the radius of the circumcircle of ABC.

   Solution: Let X be an arbitrary point on the circumcircle of ABC, and let Y be the intersection of circles BDX and CEX different from X (if the two circles are tangent, set $Y = X$).

   Then, when $X \to B$, then $Y \to D$, and similarly $X \to C$ then $Y \to E$. Moreover, as X continuously goes over the arc BC, Y must move continuously as well.

   Now, we have a lemma:

   Lemma: $Y \in DE$.
   Proof. This follows by angle chasing or by Miquel's thm applied on the triangle ADE and circles ABC, BDX, CEX. Here is the angle-chasing solution (we assume oriented angles): $\angle DYE = \angle DYX + \angle XYE = 180° - \angle XBD + 180° - \angle ECX = \angle ABX + \angle XCA = 180°$.

   The previous two facts thus give that Y traces the segment DE as X moves from B to C (and possibly something more).

   Now, we have the claim that completes the proof: There is a point X that satisfies 1) iff DE intersects the circumcircle Γ of ABC. Namely, if DE intersects Γ at Y_0, pick X that gives $Y = Y_0$. Then, we must have $X = Y$, or $X = B$, as otherwise BXD intersects ABC in three points. $X = B$ is not possible, so we must have $X = Y$, and BDX is tangent to CEX. In the other direction, if there is a point X satisfying 1), we must have $X = Y \in DE$, and then $X \in \Gamma \cap DE$.

   The final step is to prove that DE intersects ABC iff $AB \cdot AD \leq 4R^2$. Let A' be the antipode of A on Γ. It is clear that the tangent to Γ at A' is parallel to DE. Thus, we have that DE meets Γ iff $DE \cap AA'$ lies on the segment AA'. Finally, this holds iff D is on the segment AD', there D' is the foot of the perpendicular from A' onto AB, and this happens precisely when $AD \cdot AB \leq AD' \cdot AB = A'A^2 = 4R^2$.

Proposed by Aleksa Milojević.