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Algebra A

1. Given two polynomials f and g satisfying $f(x) \geq g(x)$ for all real x, a *separating line* between f and g is a line $h(x) = mx + k$ such that $f(x) \geq h(x) \geq g(x)$ for all real x. Consider the set of all possible separating lines between $f(x) = x^2 - 2x + 5$ and $g(x) = 1 - x^2$. The set of slopes of these lines is a closed interval $[a, b]$. Determine $a^4 + b^4$.

2. Let $P(x, y)$ be a polynomial with real coefficients in the variables x, y that is not identically zero. Suppose that $P(\lfloor 2a \rfloor, \lfloor 3a \rfloor) = 0$ for all real numbers a. If P has the minimum possible degree and the coefficient of the monomial y is 4, find the coefficient of x^2y^2 in P.
(The *degree* of a monomial $x^m y^n$ is $m + n$. The *degree* of a polynomial $P(x, y)$ is then the maximum degree of any of its monomials.)

3. Find the number of real solutions (x, y) to the system of equations:
$$\begin{cases} \sin(x^2 - y) = 0 \\ |x| + |y| = 2\pi \end{cases}$$

4. The set C of all complex numbers z satisfying $(z + 1)^2 = az$ for some $a \in [-10, 3]$ is the union of two curves intersecting at a single point in the complex plane. If the sum of the lengths of these two curves is ℓ, find $\lfloor \ell \rfloor$.

5. Suppose that x, y, z are nonnegative real numbers satisfying the equation
$$\sqrt{xyz} - \sqrt{(1-x)(1-y)z} - \sqrt{(1-x)y(1-z)} - \sqrt{x(1-y)(1-z)} = -\frac{1}{2}.$$
The largest possible value of $\sqrt{xy}$ equals $\frac{a+\sqrt{b}}{c}$, where a, b, and c are positive integers such that b is not divisible by the square of any prime. Find $a^2 + b^2 + c^2$.

6. Let x, y, z be positive real numbers satisfying $4x^2 - 2xy + y^2 = 64$, $y^2 - 3yz + 3z^2 = 36$, and $4x^2 + 3z^2 = 49$. If the maximum possible value of $2xy + yz - 4zx$ can be expressed as $\sqrt{n}$ for some positive integer n, find n.

7. For a positive integer $n \geq 1$, let $a_n = \lfloor \sqrt[3]{n} + \frac{1}{2} \rfloor$. Given a positive integer $N \geq 1$, let $\mathcal{F}_N$ denote the set of positive integers $n \geq 1$ such that $a_n \leq N$. Let $S_N = \sum_{n \in \mathcal{F}_N} \frac{1}{a_n^2}$. As N goes to infinity, the quantity $S_N - 3N$ tends to $\frac{a\pi^2}{b}$ for relatively prime positive integers a, b. Given that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$, find $a + b$.

8. The function f sends sequences to sequences in the following way: given a sequence $\{a_n\}_{n=0}^{\infty}$ of real numbers, f sends $\{a_n\}_{n=0}^{\infty}$ to the sequence $\{b_n\}_{n=0}^{\infty}$, where $b_n = \sum_{k=0}^{n} a_k \binom{n}{k}$ for all $n \geq 0$. Let $\{F_n\}_{n=0}^{\infty}$ be the Fibonacci sequence, defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. Let $\{c_n\}_{n=0}^{\infty}$ denote the sequence obtained by applying the function f to the sequence $\{F_n\}_{n=0}^{\infty}$ 2022 times. Find $c_5 \pmod{1000}$.

Name:

Team:

Write answers in table below:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8