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Algebra B Solutions

1. Consider the equations $x^2+y^2=16$ and $xy=\frac{9}{2}$. Find the sum, over all ordered pairs (x,y) satisfying these equations, of $|x+y|$.

   Proposed by Frank Lu

   Answer: $\boxed{20}$

   We first observe that $x=0, y=0$ are not valid values, so we can then write $x=\frac{9}{2y}$. Hence, we have that $y^2+\frac{81}{4y^2}=16$, or that y must satisfy $y^4-16y^2+\frac{81}{4}=0$. We can see that the discriminant of this equation is $16^2-81>0$, and so we have 4 distinct real solutions to this equation for y.

   Finally, observe that for each pair (x,y), we require that $(x+y)^2=x^2+y^2+2xy=16+2\cdot\frac{9}{2}=25$, or that $|x+y|=5$ for each solution. Therefore, our total sum is just the number of solutions times 5, or 20.

2. The sum
$$\sum_{m=1}^{2023}\frac{2m}{m^4+m^2+1}$$
   can be expressed as $\frac{a}{b}$ for relatively prime positive integers a,b. Find the remainder when $a+b$ is divided by 1000.

   Proposed by Sunay Joshi

   Answer: $\boxed{105}$

   If the sum runs from $m=1$ to $N-1$, then it has the closed form $\frac{N^2-N}{N^2-N+1}$, where the numerator and denominator are relatively prime. This is by telescoping: note $m^4+m^2+1=(m^2-m+1)(m^2+m+1)$, so partial fraction decomposition gives $\frac{2m}{m^4+m^2+1}=\frac{1}{m(m-1)+1}-\frac{1}{m(m+1)+1}$. Accordingly, the sum telescopes into $\frac{1}{1\cdot 0+1}-\frac{1}{(N-1)\cdot N+1}=\frac{N^2-N}{N^2-N+1}$, as claimed. Because N^2-N and N^2-N+1 differ by 1, they're relatively prime, so $a+b=2(N^2-N)+1$. Setting $N=2024$, we find the answer of 105.

3. Let a,b,c,d,e,f be real numbers such that $a^2+b^2+c^2=14$, $d^2+e^2+f^2=77$, and $ad+be+cf=32$. Find $(bf-ce)^2+(cd-af)^2+(ae-bd)^2$.

   Proposed by Sunay Joshi

   Answer: $\boxed{54}$

   Solution: Let $u=(a,b,c)$, $v=(d,e,f)$ be vectors in $\mathbb{R}^3$. Then the identity $|u\times v|^2=|u|^2|v|^2-(u\cdot v)^2$ implies that the desired expression is simply $(a^2+b^2+c^2)(d^2+e^2+f^2)-(ad+be+cf)^2$. This evaluates to $14\cdot 77-32^2=54$.

4. If θ is the unique solution in $(0,\pi)$ to the equation $2\sin(x)+3\sin\left(\frac{3x}{2}\right)+\sin(2x)+3\sin\left(\frac{5x}{2}\right)=0$, then $\cos(\theta)=\frac{a-\sqrt{b}}{c}$ for positive integers a,b,c such that a and c are relatively prime. Find $a+b+c$.

   Proposed by Ben Zenker and Nancy Xu

   Answer: $\boxed{110}$

   Using sum-to-product, we get $\sin\left(\frac{3x}{2}\right)+\sin\left(\frac{5x}{2}\right)=2\sin(2x)\cos\left(\frac{x}{2}\right)$.
   Factor out a $\sin(x)$ of the whole expression (after using double angle on $\sin(2x)$), to get:
$$\sin(x)\left(2+2\cos(x)+12\cos(x)\cos\left(\frac{x}{2}\right)\right)=0$$

$\sin(x) > 0$ in $(0, \pi)$, so we can safely ignore it. Let $u = \cos\left(\frac{x}{2}\right)$, then $\cos(x) = 2u^2 - 1$ using double angle. We now solve $2+2(2u^2-1)+12u(2u^2-1) = 0$, which becomes $6u^3+u^2-3u = 0$. The solution $u = 0$ corresponds to $x = \pi$, so we ignore it as well.
We then just need the solution to $6u^2 + u - 3 = 0$, which is $u = \frac{-1+\sqrt{73}}{12}$.
Compute $\cos(x) = 2u^2 - 1 = \frac{1-\sqrt{73}}{36}$, so $a + b + c = 1 + 73 + 36 = \boxed{110}$.

5. Let $P(x)$ be a polynomial with integer coefficients satisfying

$$(x^2 + 1)P(x - 1) = (x^2 - 10x + 26)P(x)$$

for all real numbers x. Find the sum of all possible values of $P(0)$ between 1 and 5000, inclusive.

Proposed by Sunay Joshi

Answer: $\boxed{5100}$

It is clear that the only constant solution is $P \equiv 0$, for which $P(0)$ is not in the desired range. Therefore we assume P is nonconstant in what follows. Note that since the functional equation holds for all reals, it holds for all complex numbers. Next, note that the roots of $x^2 + 1$ are $\pm i$, while the roots of $x^2 - 10x + 26$ are $\pm i + 5$. Plugging in $x = i$, we find $P(i) = 0$. Plugging in $x = i + 1$, we find $P(i + 1) = 0$. Plugging in $x = i + 2$, we find $P(i + 3) = 0$. Lastly, plugging in $x = i + 3$, we find $P(i + 4) = 0$. Since P has real coefficients, its roots also include the conjugates $-i, -i + 1, -i + 2, -i + 3, -i + 4$. Therefore $P(x)$ can be written as $P(x) = Q(x)(x^2 + 1)(x^2 - 2x + 2)(x^2 - 4x + 5)(x^2 - 6x + 10)(x^2 - 8x + 17)$. We now claim that $Q(x)$ is a nonzero constant. Plugging our expression for P into our functional equation, we find $Q(x - 1) = Q(x)$ for all x, hence $Q(x) \equiv c \neq 0$ is a constant.

To finish, set $x = 0$ to find $P(x) = 1700c$. The only integer multiples of 1700 between 1 and 5000 are 1700 and 3400, hence our answer is $1700 + 3400 = 5100$.

6. The set of real values of a such that the equation $x^4 - 3ax^3 + (2a^2 + 4a)x^2 - 5a^2x + 3a^2$ has exactly two nonreal solutions is the set of real numbers between x and y, where $x < y$. If $x + y$ can be written as $\frac{m}{n}$ for relatively prime positive integers m, n, find $m + n$.

Proposed by Frank Lu

Answer: $\boxed{8}$

First, we consider trying to factor this into quadratics. Notice that this equals

$$x^4 - 3tx^3 + (2t^2 - 2t)x + t^2x - 3t^2 = (x^2 - tx + t)(x^2 - 2tx + 3t).$$

Therefore, to have two nonreal solutions, one of the discriminants of the quadratics needs to be negative, and the other is nonnegative. In particular, it follows that we need $t^2 - 4t < 0$ and $4t^2 - 12t \geq 0$ or $t^2 - 4t \geq 0$ and $4t^2 - 12t < 0$. For the former to hold, notice that we need $0 < t < 4$, but $t > 3$. The latter cannot hold, however: $t^2 - 4t \geq 0$ implies that $t \geq 4$ or $t \leq 0$, but $4t^2 - 12t < 0$ implies that $0 < t < 3$. Therefore, we see that $a = 3, b = 4$, and $a + b = 7 = 7/1$. Our answer is thus $7 + 1 = 8$.

7. Compute $\left\lfloor \sum_{k=0}^{10} \left(3 + 2\cos\left(\frac{2\pi k}{11}\right)\right)^{10} \right\rfloor \pmod{100}$.

Proposed by Sunay Joshi and Ben Zenker

Answer: $\boxed{91}$

Let $n = 10$. We claim that the sum equals

$$(n + 1) \sum_{k=0}^{\lfloor n/2 \rfloor} 3^{n-2k} \binom{n}{2k} \binom{2k}{k} \tag{1}$$

Let $\omega = \exp(2\pi i/(n+1))$. The summand is $(\omega^k + \omega^{-k} + 3)^n$, which by the multinomial expansion equals $\sum_{a+b+c=n} \binom{n}{a,b,c} 3^c \omega^{k(a-b)}$. Since $0 \le |a-b| < n+1$, $\sum_{k=0}^{n} = (n+1)\mathbf{1}_{a=b}$. Therefore the sum becomes

$$(n+1)\sum_{a+b+c=n}\binom{n}{a,b,c}3^{n-a-b}\mathbf{1}_{a=b} = (n+1)\sum_{a=0}^{n}3^{n-2a}\binom{n}{a,a,n-2a} \tag{2}$$

$$= (n+1)\sum_{a=0}^{\lfloor n/2\rfloor}3^{n-2a}\frac{n!}{a!a!(n-2a)!} \tag{3}$$

$$= (n+1)\sum_{a=0}^{\lfloor n/2\rfloor}3^{n-2a}\binom{n}{2a}\binom{2a}{a}, \tag{4}$$

as claimed.

The desired remainder is therefore

$$11\cdot\left[3^{10}\binom{10}{0}\binom{0}{0}+3^8\binom{10}{2}\binom{2}{1}+3^6\binom{10}{4}\binom{4}{2}+3^4\binom{10}{6}\binom{6}{3}+3^2\binom{10}{8}\binom{8}{4}+3^0\binom{10}{10}\binom{10}{5}\right] \tag{5}$$

$$\equiv 11\cdot\left[3^{10}+3^8\cdot 90+3^6\cdot 10\cdot 6+3^4\cdot 10\cdot 20+3^2\cdot 45\cdot 70+52\right] \tag{6}$$

$$\equiv 91 \pmod{100} \tag{7}$$

8. A polynomial $p(x) = \sum_{j=1}^{2n-1} a_j x^j$ with real coefficients is called *mountainous* if $n \ge 2$ and there exists a real number k such that the polynomial's coefficients satisfy $a_1 = 1$, $a_{j+1} - a_j = k$ for $1 \le j \le n-1$, and $a_{j+1} - a_j = -k$ for $n \le j \le 2n-2$; we call k the *step size* of $p(x)$. A real number k is called *good* if there exists a mountainous polynomial $p(x)$ with step size k such that $p(-3) = 0$. Let S be the sum of all good numbers k satisfying $k \ge 5$ or $k \le 3$. If $S = \frac{b}{c}$ for relatively prime positive integers b, c, find $b + c$.

   Proposed by Sunay Joshi

   Answer: 101

   We claim that the only good values of k are $k = \frac{7}{3}$ and $\frac{61}{12}$, corresponding to $n = 2$ and $n = 3$ respectively. This yields $S = \frac{89}{12}$ and an answer of 101.

   To see this, note that a generic mountainous polynomial $p(x)$ can be written as

$$p(x) = (1-k)\frac{x^{2n}-x}{x-1} + kx\frac{(x^n-1)^2}{(x-1)^2}$$

   if $x \ne 1$. This follows from the observation that $\frac{x^{2n}-x}{x-1} = x + x^2 + \ldots + x^{2n-1}$ and $\frac{(x^n-1)^2}{(x-1)^2} = (x^{n-1} + x^{n-2} + \ldots + 1)^2 = x + 2x^2 + \ldots + nx^n + (n-1)x^{n+1} + \ldots + x^{2n-2}$. Hence $p(x) = 0$ implies that $(1-k)\frac{x^{2n}-x}{x-1} + kx\frac{(x^n-1)^2}{(x-1)^2} = 0$. Rearranging and solving for k, we find

$$k = 1 - \frac{x^n + \frac{1}{x^n} - 2}{x^{n-1} + \frac{1}{x^{n-1}} - 2}$$

   As $n \to \infty$, $k = k(n)$ tends to $1 - x$. In our case $x = -3$, so the limit equals 4. It follows that there are only finitely many n such that $|k-4| \ge 1$. Calculating $k(n)$ for $n = 2, 3, 4$, we find $k(2) = 7/3$, $k(3) = 61/12$.

We claim that for $n \geq 4$, $|k(n) - 4| < 1$, so that $n = 2, 3$ are the only valid cases. Note that

$$|k(n) - 4| = \left| \frac{8 + \frac{8}{(-3)^n}}{(-3)^{n-1} + \frac{1}{(-3)^{n-1}} - 2} \right|$$

We split into the cases when n is even ($n \geq 4$) and n is odd ($n \geq 5$).

If n is even, then

$$|k(n) - 4| = \frac{8 + \frac{8}{3^n}}{3^{n-1} + \frac{1}{3^{n-1}} + 2}$$

The inequality $|k(n) - 4| < 1$ is equivalent to $\frac{1}{3}3^{2n} - 6 \cdot 3^n - 5 > 0$, i.e. $\frac{1}{3}x^2 - 6x - 5 > 0$ for $x \geq 81$, which is true.

If n is odd, then

$$|k(n) - 4| = \frac{8 - \frac{8}{3^n}}{3^{n-1} + \frac{1}{3^{n-1}} - 2}$$

The inequality $|k(n) - 4| < 1$ is equivalent to $\frac{1}{3}3^{2n} - 10 \cdot 3^n + 11 > 0$, i.e. $\frac{1}{3}x^2 - 10x + 11 > 0$ for $x \geq 243$, which is true. The result follows.