P U M a C

Combinatorics A

1. In the country of PUMaC-land, there are 5 villages and 3 cities. Vedant is building roads between the 8 settlements according to the following rules:

   a) There is at most one road between any two settlements;

   b) Any city has exactly three roads connected to it;

   c) Any village has exactly one road connected to it;

   d) Any two settlements are connected by a path of roads.

   In how many ways can Vedant build the roads?

2. Ten evenly spaced vertical lines in the plane are labeled $\ell_1, \ell_2, \ldots, \ell_{10}$ from left to right. A set $\{a, b, c, d\}$ of four distinct integers $a, b, c, d \in \{1, 2, \ldots, 10\}$ is *squarish* if some square has one vertex on each of the lines ℓ_a, ℓ_b, ℓ_c, and ℓ_d. Find the number of squarish sets.

3. Randy has a deck of 29 distinct cards. He chooses one of the 29! permutations of the deck and then repeatedly rearranges the deck using that permutation until the deck returns to its original order for the first time. What is the maximum number of times Randy may need to rearrange the deck?

4. Let C_n denote the n-dimensional unit cube, consisting of the 2^n points

   $$\{(x_1, x_2, \ldots, x_n) | x_i \in \{0, 1\} \text{ for all } 1 \leq i \leq n\}.$$

   A tetrahedron is *equilateral* if all six side lengths are equal. Find the smallest positive integer n for which there are four distinct points in C_n that form a non-equilateral tetrahedron with integer side lengths.

5. An *n-folding process* on a rectangular piece of paper with sides aligned vertically and horizontally consists of repeating the following process n times:

   - Take the piece of paper and fold it in half vertically (choosing to either fold the right side over the left, or the left side over the right).
   - Rotate the paper 90° degrees clockwise.

   A 10-folding process is performed on a piece of paper, resulting in a 1-by-1 square base consisting of many stacked layers of paper. Let $d(i, j)$ be the Euclidean distance between the center of the ith square from the top and the center of the jth square from the top before the paper was folded. Determine the maximum possible value of $\sum_{i=1}^{1023} d(i, i+1)$.

6. Fine Hall has a broken elevator. Every second, it goes up a floor, goes down a floor, or stays still. You enter the elevator on the lowest floor, and after 8 seconds, you are again on the lowest floor. If every possible such path is equally likely to occur, the probability you experience no stops is $\frac{a}{b}$, where a, b are relatively prime positive integers. Find $a + b$.

7. Kelvin has a set of eight vertices. For each pair of distinct vertices, Kelvin independently draws an edge between them with probability $p \in (0, 1)$. A set S of four distinct vertices is called *good* if there exists an edge between v and w for all $v, w \in S$ with $v \neq w$. The variance of the number of good sets can be expressed as a polynomial $f(p)$ in the variable p. Find the sum of the absolute values of the coefficients of $f(p)$.

   (The *variance* of random variable X is defined as $\mathbb{E}[X^2] - \mathbb{E}[X]^2$.)

8. A permutation $\pi : \{1, 2, \ldots, N\} \to \{1, 2, \ldots, N\}$ is *very odd* if the smallest positive integer k such that $\pi^k(a) = a$ for all $1 \leq a \leq N$ is odd, where π^k denotes π composed with itself k times. Let $X_0 = 1$, and for $i \geq 1$, let X_i be the fraction of all permutations of $\{1, 2, \ldots, i\}$ that are very odd. Let $\mathcal{S}$ denote the set of all ordered 4-tuples (A, B, C, D) of nonnegative integers such that $A + B + C + D = 2023$. Find the last three digits of the integer

$$2023 \sum_{(A,B,C,D) \in \mathcal{S}} X_A X_B X_C X_D.$$

Name:

Team:

Write answers in table below:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8