P U M . . C

Combinatorics A Solutions

1. Alien Connor starts at $(0,0)$ and walks around on the integer lattice. Specifically, he takes one step of length one in a uniformly random cardinal direction every minute, unless his previous four steps were all in the same direction in which case he randomly picks a new direction to step in. Every time he takes a step, he leaves toxic air on the lattice point he just left, and the toxic cloud remains there for 150 seconds. After taking 5 steps in total, the probability that he has not encountered his own toxic waste can be written as $\frac{a}{b}$ for relatively prime positive integers a, b. Find $a+b$.

   Proposed by Ben Zenker

   Answer: $\boxed{505}$
   Due to parity, we can see that the only way he can encounter his own toxic waste is by walking directly backwards. The toxic waste stays in the air for 2 full step sizes, but disappears after 3, and there's no way to take two more steps and return to where you started.
   First, suppose his first four steps are all in the same direction, which happens with probability $\frac{1}{4^3}$. Then, the probability he avoids his own toxic waste with his last step is $\frac{2}{3}$, contributing a probability of $\frac{2}{3} \cdot \frac{1}{4^3}$. Otherwise, we see the probability he makes it four steps without hitting is own toxic waste while also not going the same direction every step is $\left(\frac{3}{4}\right)^3 - \frac{1}{4^3} = \frac{13}{32}$. Conditioned on this, we see the probability his last step also avoids the toxic air is again $\frac{3}{4}$. Thus, our final answer is $\frac{2}{3} \cdot \frac{1}{4^3} + \frac{13}{32} \cdot \frac{3}{4} = \frac{4}{3\cdot 128} + \frac{117}{3\cdot 128} = \frac{121}{384}$, giving a final answer of $121 + 384 = 505$.

2. Let $\oplus$ denote the xor binary operation. Define $x \star y = (x+y) - (x \oplus y)$. Compute

$$\sum_{k=1}^{63} (k \star 45).$$

   (*Remark:* The xor operator works as follows: when considered in binary, the kth binary digit of $a \oplus b$ is 1 exactly when the kth binary digits of a and b are different. For example, $5 \oplus 12 = 0101_2 \oplus 1100_2 = 1001_2 = 9$.)

   Proposed by Julian Shah

   Answer: $\boxed{2880}$
   Solution 1: Consider pairing the kth term with the $(63-k)$th term:

$$(k \star 45) + ((63-k) \star 45) = 63 + 2 \cdot 45 - [k \oplus 45 + (63-k) \oplus 45]$$

   k and $63-k$ differ in every binary digit, so the values of $k \oplus 45$ and $(63-k) \oplus 45$ will be completely complementary; hence, they add to 63 (when performing the addition $k \oplus 45$, switching a 0 to a 1 in k's representation will switch the result of that output digit).

   So, when we pair k and $63-k$, the result is $2 \cdot 45$. This means $\sum\limits_{k=0}^{63} (k \star 45) = 64 \cdot 45$, but $(0 \star 45) = 45 - 45 = 0$, so our final answer is actually $64 \cdot 45 = \boxed{2880}$.
   Solution 2: For any x and y, we can obtain the relationship: $(x \oplus y) + 2(x \& y) = x + y$, where $x \& y$ is the bitwise-*and* operator: $x \& y$ has a 1 in a place only if both x and y do.
   This occurs since $x \oplus y$ returns 0 in all places where x and y are both 1, whereas under normal addition this *should* contribute 2 times that place value. Adding $2(x\&y)$ covers this difference. Therefore, $x \star y = (x+y) - (x \oplus y) = 2(x \& y)$.
   Again, note that $((63-k)\&45) + (k\&45) = 45$. Collectively, k and $63-k$ will 'select' all the 1's places of 45, so adding them will return exactly 45.
   Using the same trick as earlier, $\sum\limits_{k=0}^{63} (k\&45) = 32 \cdot 45$, so our answer is twice this, $2 \cdot 32 \cdot 45 = \boxed{2880}$.

3. The integers from 1 to 25, inclusive, are randomly placed into a 5 by 5 grid such that in each row, the numbers are increasing from left to right. If the columns from left to right are numbered 1, 2, 3, 4, and 5, then the expected column number of the entry 23 can be written as $\frac{a}{b}$ where a and b are relatively prime positive integers. Find $a+b$.

   Proposed by Rishi Dange

   Answer: $\boxed{17}$

   The answer is $\frac{14}{3}$. We proceed using casework, seeing pretty easily that 23 cannot be in columns 1 or 2.

   Case 1: 23 is in column 5 In a given row, by putting 23 as the rightmost item, there are now $\binom{22}{4}$ ways to fill in the rest of the row. This makes for a probability of 5 rows $\times \frac{\binom{22}{4}}{\binom{25}{5}}$.

   Case 2: 23 is in column 4 The right two elements in the row with 23 are either 23 and 24 or 23 and 25, which makes the "rows" coefficient above $5 \times 2 = 10$. Thus we now have a probability of $10 \times \frac{\binom{22}{3}}{\binom{25}{5}}$.

   Case 3: 23 is in column 3 The right three elements must be 23, 24, and 25, so the coefficient is 5 again to represent the number of possible rows. Thus we now have a probability of $5 \times \frac{\binom{22}{2}}{\binom{25}{5}}$.

   Computing the expected value using these probabilities (quite a bit of stuff cancels out), we find the expected value to be $\frac{14}{3}$.

4. A sequence of integers $a_1, a_2, \ldots, a_n$ is said to be *sub-Fibonacci* if $a_1 = a_2 = 1$ and $a_i \leq a_{i-1} + a_{i-2}$ for all $3 \leq i \leq n$. How many sub-Fibonacci sequences are there with 10 terms such that the last two terms are both 20?

   Proposed by Daniel Carter

   Answer: $\boxed{238}$

   The number of sequences of length 10 that end in 20, 20 is just the number of sequences of length 9 which end in 20, since it is impossible for it to be the case that $a_8 < 0$ and $a_9 = 20$, as the seventh Fibonacci number (i.e. the maximum possible value for a_7) is only 13.

   Let F_n be the Fibonacci numbers, where $F_1 = F_2 = 1$. Suppose we chose the maximum value $a_{i-1} + a_{i-2}$ for every term a_i in our sequence except for some a_j, which we made k less than the maximum possible value. Then $a_n = F_n - kF_{n-j+1}$. This works similarly if we make multiple terms less than their maximum; if we define $d_i = a_i - a_{i-1} - a_{i-2}$, then we find $a_n = F_n - \sum_{i=3}^{n} d_i F_{n-i+1}$. Since $F_9 = 34$, the question is equivalent to asking for the number of choices of d_i which make $\sum_{i=3}^{9} d_i F_{10-i} = 14$.

   In order to compute this, let's define $f(k,t)$ to be the number of choices of d_i such that $\sum_{i=1}^{t} d_i F_i = k$. By convention, $f(0,t) = 1$ for all t and $f(k,t) = 0$ if k is negative. We are looking for $f(14,7)$. We have $f(k,t) = f(k,t-1) + f(k-F_t,t)$, i.e. we either stop increasing d_t and move on to smaller t or increment d_t. With this recurrence, we can quickly fill up a table of values for f until we hit $f(14,7)$, which we find to be 238.

5. There are n assassins numbered from 1 to n, and all assassins are initially alive. The assassins play a game in which they take turns in increasing order of number, with assassin 1 getting the first turn, then assassin 2, etc., with the order repeating after assassin n has gone; if an assassin is dead when their turn comes up, then their turn is skipped and it goes to the next assassin in line. On each assassin's turn, they can choose to either kill the assassin who would otherwise move next or to do nothing. Each assassin will kill on their turn unless the only option for guaranteeing their own survival is to do nothing. If there are 2023 assassins at the

start of the game, after an entire round of turns in which no one kills, how many assassins must remain?

Proposed by Ben Lemkin

Answer: $\boxed{1023}$

We show by induction that number of the form $n = 2^k - 1$ are stable, being no one shoots, while all others are not. $n = 1, 2$ are trivial; for $n = 3$, we observe it's stable for the following reason; if person 1 shoots person 2, then person 3 will kill person 1, so to avoid dying person 1 will not shoot; by symmetry, this means none of them will shoot. Indeed, note by symmetry that we only need to assess the first person's behavior, as if person 1 shoots then it's not stable, while if person 1 does not shoot then it is stable. With the base case proven, suppose $n = 2^k - 1$ is stable. Consider $n = 2^k$; the first person will shoot the second person, because then there are $2^k - 1$ people remaining, and by assumption no one will shoot in this scenario. Similarly, for $n = 2^k + 1$, if the first person shoots the second person, then 2^k people will remain standing, whence the third person will proceed to shoot the fourth person after which $2^k - 1$ people are left and everyone ceases. Thus, the first person will shoot the second person as long as the first person is not also the fourth person (meaning $4 \equiv 1 \pmod{2^k - 1}$). Continuing like so, we see that for $n = 2^k + a$ for $a \geq 0$ that person 1 will shoot, then person 3 will shoot, etc., all the way up to person $2(a+1) - 1$ shooting person $2(a+1)$, after which there are $2^k - 1$ people left and no one else shoots. This sequence of events will hold up until the point when $2(a+1) \equiv 1 \pmod{2^k + a}$, because that would means person 1 gets shot; the smallest a this holds for, and thus the next smallest stable position, is $a = 2^k - 1$, implying $n = 2^{k+1} - 1$. Thus, by induction, stable n are of the form $2^k - 1$, and our answer is 1023.

6. For a positive integer n, let P_n be the set of sequences of $2n$ elements, each 0 or 1, where there are exactly n 1's and n 0's. I choose a sequence uniformly at random from P_n. Then, I partition this sequence into maximal blocks of consecutive 0's and 1's. Define $f(n)$ to be the expected value of the sum of squares of the block lengths of this uniformly random sequence. What is the largest integer value that $f(n)$ can take on?

   Proposed by Kevin Ren

   Answer: $\boxed{121}$

   It's easier to compute the expected value of $\sum \binom{a}{2}$, where the a's are the block lengths. Note that
$$\mathbb{E}\left[\sum a^2\right] = \mathbb{E}\left[\sum 2\binom{a}{2} + a\right] = 2\mathbb{E}\left[\binom{a}{2}\right] + 2n,$$
   since the sum of block lengths is clearly $2n$. Hence, it suffices to show $\mathbb{E}\left[\sum \binom{a}{2}\right] = 2 \cdot \frac{n(n-1)}{n+2}$. Note that $\sum \binom{a}{2}$ equals the number of pairs of indices (i, j) such that (i, j) belong to the same block. We take advantage of this by instead computing the expectation that (i, j) belong to the same block, and sum over (i, j) (this amounts to a change of summation). The probability that (i, j) belong to the same block is $2 \cdot \binom{2n-|j-i+1|}{n} / \binom{2n}{n}$. And then by various binomial identities,
$$\sum_{i<j} 2 \cdot \binom{2n - |j-i+1|}{n} = \sum_{k=2}^{2n} 2(2n-k+1)\binom{2n-k}{n} = \sum_{k=2}^{2n} 2(n+1)\binom{2n-k+1}{n+1}$$
$$= 2(n+1)\binom{2n}{n+2}.$$
   Finally, $\mathbb{E}\left[\sum \binom{a}{2}\right] = 2(n+1)\binom{2n}{n+2} / \binom{2n}{n} = 2 \cdot \frac{n(n-1)}{n+2}$, as desired. This gives the actual desired expected value as $\frac{6n^2}{n+2}$. In order for this to be an integer, note that it equals $\frac{6(n-2)(n+2)}{n+2} + \frac{24}{n+2}$,

which is an integer precisely when $n+2|24$. It is evidently maximized for $n=22$, giving an answer of $\frac{6\cdot 22^2}{24}=121$.

7. A utility company is building a network to send electricity to fifty houses, with addresses $0,1,2,\ldots,49$. The power center only connects directly to house 0, so electricity reaches all other houses through a system of wires that connects specific pairs of houses. To save money, the company only lays wires between as few pairs of distinct houses as possible; additionally, two houses with addresses a and b can only have a wire between them if at least one of the following three conditions is met:

   - 10 divides both a and b.
   - $\left\lfloor \frac{b}{10} \right\rfloor \equiv \left\lfloor \frac{a}{10} \right\rfloor \pmod 5$.
   - $\left\lceil \frac{b}{10} \right\rceil \equiv \left\lceil \frac{a}{10} \right\rceil \pmod 5$.

   Letting N be the number of distinct ways such a wire system can be configured so that every house receives electricity, find the remainder when N is divided by 1000.

   Proposed by Austen Mazenko

   Answer: $\boxed{810}$

   Interpreting the network as a graph G, with houses as vertices and wires as edges, we see that the conditions reduce to the graph being comprised of five K_{11} cliques and one K_5 clique. Specifically, the groups of houses $\{0,...,10\},\{10,...,20\},\{20,...,30\},\{30,...,40\}$, and $\{40,...,49,0\}$ are all fully connected within themselves forming K_{11} cliques, and the vertices $\{0,10,20,30,40\}$ form a K_5 clique. Now, a connected subgraph with a minimal number of edges, which is precisely what any valid configuration of the wire network is, is by definition just a spanning tree of the graph. Thus, we seek the number of spanning trees. Call the vertices $0,10,20,30,40$ *connectors*, because any path in a spanning tree between two vertices in different K_{11} cliques must include a connector. Let C be an auxiliary K_5 graph with vertices that are phantom copies of the connectors. Recall that on a tree, there is a unique shortest path between any two vertices; the crucial observation is that any spanning tree $\mathcal{T}$ on G can be associated with a spanning tree $f(\mathcal{T})$ on C in which two connectors are adjacent if the shortest path between those connectors on $\mathcal{T}$ doesn't pass through another connector.

   To count the number of spanning trees on G, we will split into casework based on which spanning tree on C they correspond to (namely, based on the image of the map f taking spanning trees on G to spanning trees on C). Call two connectors *near* if they are in a K_{11} clique together, and *far* otherwise (e.g. 0 is near 10 and 40 and far from 20 and 30). Note that two far connectors will share an edge in $\mathcal{T}$ iff they do in $f(\mathcal{T})$. Next, if two near connectors share an edge in $f(\mathcal{T})$, then the shortest path in $\mathcal{T}$ connecting them lies in the K_{11} containing both of them. Moreover, because these connectors silo off this K_{11} from the rest of G, we see that $\mathcal{T}$ forms a spanning tree on this K_{11}; the number of ways this can happen is 11^9 by Cayley's formula. Finally, if two near connectors don't share an edge in $f(\mathcal{T})$, then there can be no path between them in the K_{11} containing both of them. However, every vertex in this K_{11} must be path-connected to one of these connectors via $\mathcal{T}$, so the restriction of $\mathcal{T}$ to this K_{11} looks like two disjoint spanning trees, one for each of the two connectors. We claim that the number of ways to choose two disjoint spanning trees on K_{11} rooted at two fixed vertices v_1,v_2 (the connectors) is $2\cdot 11^8$. Note that every such pair of disjoint spanning trees can be uniquely associated with a single spanning tree on K_{11} by adding the edge v_1v_2. Thus, it's equivalent to counting the number of spanning trees on K_{11} containing a fixed edge. Now, if we pick a spanning tree at random, by Cayley's formula there are 11^9 to pick from, and by symmetry the probability a given edge is in the spanning tree is $\frac{10}{\binom{11}{2}}=\frac{2}{11}$ (the number of edges in each spanning tree divided by the number of edges in K_{11}). Hence, there are $\frac{2}{11} \cdot 11^9 = 2 \cdot 11^8$ ways to pick two disjoint spanning trees rooted at two fixed vertices, as claimed.

Putting this all together, for a given spanning tree on C, the number of spanning trees $\mathcal{T}$ of G that map to it will be $(11^9)^a \cdot (2 \cdot 11^8)^{5-a}$, where a is the number of edges between near connectors in the spanning tree on C. Doing casework on the spanning trees of $C \simeq K_5$, we see that $a = 4$ in 5 of them, $a = 3$ in $2 \cdot 5 + 4 \cdot 5 = 30$, $a = 2$ in $4 \cdot 5 + 7 \cdot 5 = 55$, $a = 1$ in 30, and $a = 0$ in 5 of them.

In conclusion, the total number of configurations is

$$N = 5 \cdot 11^{36} \cdot 2 \cdot 11^8 + 30 \cdot 11^{27} \cdot 2^2 \cdot 11^{16} + 55 \cdot 11^{18} \cdot 2^3 \cdot 11^{24} + 30 \cdot 11^9 \cdot 2^4 \cdot 11^{32} + 5 \cdot 2^5 \cdot 11^{40},$$

and taking modulo 1000 gives $410 + 720 + 240 + 280 + 160 = 810$.

8. A spider is walking on the boundary of equilateral triangle $\triangle ABC$ (vertices labelled in counterclockwise order), starting at vertex A. Each second, she moves to one of her two adjacent vertices with equal probability. The *windiness* of a path that starts and ends at A is the net number of counterclockwise revolutions made. For example, the windiness of the path $ABCA$ is 1, and the windiness of the path $ABCACBACBA$ is -1. What is the remainder modulo 1000 of the sum of the squares of the windiness values taken over all possible paths that end back at vertex A after 2025 seconds?

   Proposed by Atharva Pathak

   Answer: $\boxed{50}$

   Let $3n = 2025$. We seek $S = \sum_{k=0}^{n} \binom{3n}{3k}(2k-n)^2$. Expanding, this equals

$$S = \sum_{k=0}^{n} \binom{3n}{3k}(4k^2 - 4nk + n^2) \tag{1}$$

   Let $p(x) = (1+x)^{3n}$. Note that $p(x) = \sum_{\ell=0}^{3n} \binom{3n}{\ell} x^\ell$. Note that $xp'(x) = (3n)x(1+x)^{3n-1} = \sum_{\ell=0}^{3n} \binom{3n}{\ell} \ell x^\ell$. Note that $x^2 p''(x) = (3n)(3n-1)x^2(1+x)^{3n-2} = \sum_{\ell=0}^{3n} \binom{3n}{\ell} \ell(\ell-1)x^\ell$. (Here, $q'(x)$ denotes the formal derivative of the polynomial q, defined by $q'(x) = ax^{a-1}$ for $q(x) = x^a$ and extended linearly.)

   Note that $4 \cdot \frac{1}{9}[\ell(\ell-1)+\ell] - 4n \cdot \frac{1}{3}\ell + n^2 = \frac{4}{9}\ell^2 - \frac{4n}{3}\ell + n^2$. For $\ell = 3k$, this equals $4k^2 - 4nk + n^2$. Let $\zeta = \exp(2\pi i/3)$. We apply the roots of unity filter to obtain

$$S = \frac{1}{3}\sum_{j=0}^{2}[\frac{4}{9}(x^2p''(x) - xp'(x)) - \frac{4n}{3}xp'(x) + n^2 p(x)](\zeta^j) \tag{2}$$

   where we evaluate the summand at $x = \zeta^j$. The summand simplifies to the expression

$$\frac{1}{3}n(1+x)^{3n-2}(3n(x-1)^2 + 4x) \tag{3}$$

   For $x = 1$, the summand is clearly $\frac{1}{3}n2^{3n-2} \cdot 4 = \frac{1}{3}n8^n$. For $x = \zeta$, the summand is

$$\frac{1}{3}n(1+\zeta)^{3n-2}(3n(\zeta-1)^2 + 4\zeta) = \frac{1}{3}n(-1)^n(-9n+4) \tag{4}$$

   using the fact that $1 + \zeta = -\zeta^2$ and $(\zeta - 1)^2 = -3\zeta$. By symmetry the same holds for $x = \zeta^2$. It follows that the sum is given by

$$S = \frac{1}{3}[\frac{1}{3}n8^n + \frac{2}{3}(-1)^n(-9n^2 + 4n)] = \frac{1}{9}n8^n + \frac{2}{9}(-1)^n n(4 - 9n)$$

   We set $n = 2025/3 = 675$ and compute mod 1000 to obtain the desired answer.