# Combinatorics A

1. Alien Connor starts at $(0, 0)$ and walks around on the integer lattice. Specifically, he takes one step of length one in a uniformly random cardinal direction every minute, unless his previous four steps were all in the same direction in which case he randomly picks a new direction to step in. Every time he takes a step, he leaves toxic air on the lattice point he just left, and the toxic cloud remains there for 150 seconds. After taking 5 steps in total, the probability that he has not encountered his own toxic waste can be written as $\frac{a}{b}$ for relatively prime positive integers $a, b$. Find $a + b$.

2. Let $\oplus$ denote the xor binary operation. Define $x \star y = (x + y) - (x \oplus y)$. Compute
$$\sum_{k=1}^{63}(k \star 45).$$
(*Remark:* The xor operator works as follows: when considered in binary, the $k$th binary digit of $a \oplus b$ is 1 exactly when the $k$th binary digits of $a$ and $b$ are different. For example, $5 \oplus 12 = 0101_2 \oplus 1100_2 = 1001_2 = 9$.)

3. The integers from 1 to 25, inclusive, are randomly placed into a 5 by 5 grid such that in each row, the numbers are increasing from left to right. If the columns from left to right are numbered 1, 2, 3, 4, and 5, then the expected column number of the entry 23 can be written as $\frac{a}{b}$ where $a$ and $b$ are relatively prime positive integers. Find $a + b$.

4. A sequence of integers $a_1, a_2, \ldots, a_n$ is said to be *sub-Fibonacci* if $a_1 = a_2 = 1$ and $a_i \leq a_{i-1} + a_{i-2}$ for all $3 \leq i \leq n$. How many sub-Fibonacci sequences are there with 10 terms such that the last two terms are both 20?

5. There are $n$ assassins numbered from 1 to $n$, and all assassins are initially alive. The assassins play a game in which they take turns in increasing order of number, with assassin 1 getting the first turn, then assassin 2, etc., with the order repeating after assassin $n$ has gone; if an assassin is dead when their turn comes up, then their turn is skipped and it goes to the next assassin in line. On each assassin's turn, they can choose to either kill the assassin who would otherwise move next or to do nothing. Each assassin will kill on their turn unless the only option for guaranteeing their own survival is to do nothing. If there are 2023 assassins at the start of the game, after an entire round of turns in which no one kills, how many assassins must remain?

6. For a positive integer $n$, let $P_n$ be the set of sequences of $2n$ elements, each 0 or 1, where there are exactly $n$ 1's and $n$ 0's. I choose a sequence uniformly at random from $P_n$. Then, I partition this sequence into maximal blocks of consecutive 0's and 1's. Define $f(n)$ to be the expected value of the sum of squares of the block lengths of this uniformly random sequence. What is the largest integer value that $f(n)$ can take on?

7. A utility company is building a network to send electricity to fifty houses, with addresses $0, 1, 2, \ldots, 49$. The power center only connects directly to house 0, so electricity reaches all other houses through a system of wires that connects specific pairs of houses. To save money, the company only lays wires between as few pairs of distinct houses as possible; additionally, two houses with addresses $a$ and $b$ can only have a wire between them if at least one of the following three conditions is met:
   - 10 divides both $a$ and $b$.

- $\left\lfloor \frac{b}{10} \right\rfloor \equiv \left\lfloor \frac{a}{10} \right\rfloor \pmod{5}$.
- $\left\lceil \frac{b}{10} \right\rceil \equiv \left\lceil \frac{a}{10} \right\rceil \pmod{5}$.

Letting $N$ be the number of distinct ways such a wire system can be configured so that every house receives electricity, find the remainder when $N$ is divided by 1000.

8. A spider is walking on the boundary of equilateral triangle $\triangle ABC$ (vertices labelled in counterclockwise order), starting at vertex $A$. Each second, she moves to one of her two adjacent vertices with equal probability. The *windiness* of a path that starts and ends at $A$ is the net number of counterclockwise revolutions made. For example, the windiness of the path $ABCA$ is 1, and the windiness of the path $ABCACBACBA$ is $-1$. What is the remainder modulo 1000 of the sum of the squares of the windiness values taken over all possible paths that end back at vertex $A$ after 2025 seconds?

**Name:**

**Team:**

**Write answers in table below:**

| Q1 | Q2 | Q3 | Q4 | Q5 | Q6 | Q7 | Q8 |
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