PUMaC

Combinatorics B

1. Betty has a 4-by-4 square box of chocolates. Every time Betty eats a chocolate, she picks one from a row with the greatest number of remaining chocolates. In how many ways can Betty eat 5 chocolates from her box, where order matters?

2. The *base factorial* number system is a unique representation for positive integers where the nth digit from the right ranges from 0 to n inclusive and has place value $n!$ for all $n \geq 1$. For instance, 71 can be written in base factorial as $2321_! = 2 \cdot 4! + 3 \cdot 3! + 2 \cdot 2! + 1 \cdot 1!$. Let $S_!(n)$ be the base 10 sum of the digits of n when n is written in base factorial. Compute $\sum_{n=1}^{700} S_!(n)$ (expressed in base 10).

3. In the country of PUMaC-land, there are 5 villages and 3 cities. Vedant is building roads between the 8 settlements according to the following rules:

   a) There is at most one road between any two settlements;

   b) Any city has exactly three roads connected to it;

   c) Any village has exactly one road connected to it;

   d) Any two settlements are connected by a path of roads.

   In how many ways can Vedant build the roads?

4. Ten evenly spaced vertical lines in the plane are labeled $\ell_1, \ell_2, \ldots, \ell_{10}$ from left to right. A set $\{a, b, c, d\}$ of four distinct integers $a, b, c, d \in \{1, 2, \ldots, 10\}$ is *squarish* if some square has one vertex on each of the lines ℓ_a, ℓ_b, ℓ_c, and ℓ_d. Find the number of squarish sets.

5. Randy has a deck of 29 distinct cards. He chooses one of the 29! permutations of the deck and then repeatedly rearranges the deck using that permutation until the deck returns to its original order for the first time. What is the maximum number of times Randy may need to rearrange the deck?

6. Let C_n denote the n-dimensional unit cube, consisting of the 2^n points

   $$\{(x_1, x_2, \ldots, x_n) | x_i \in \{0, 1\} \text{ for all } 1 \leq i \leq n\}.$$

   A tetrahedron is *equilateral* if all six side lengths are equal. Find the smallest positive integer n for which there are four distinct points in C_n that form a non-equilateral tetrahedron with integer side lengths.

7. An *n-folding process* on a rectangular piece of paper with sides aligned vertically and horizontally consists of repeating the following process n times:

   - Take the piece of paper and fold it in half vertically (choosing to either fold the right side over the left, or the left side over the right).
   - Rotate the paper 90° degrees clockwise.

   A 10-folding process is performed on a piece of paper, resulting in a 1-by-1 square base consisting of many stacked layers of paper. Let $d(i, j)$ be the Euclidean distance between the center of the ith square from the top and the center of the jth square from the top when the paper is unfolded. Determine the maximum possible value of $\sum_{i=1}^{1023} d(i, i+1)$.

8. Fine Hall has a broken elevator. Every second, it goes up a floor, goes down a floor, or stays still. You enter the elevator on the lowest floor, and after 8 seconds, you are again on the lowest floor. If every possible such path is equally likely to occur, the probability you experience no stops is $\frac{a}{b}$, where a, b are relatively prime positive integers. Find $a + b$.

Name:

Team:

Write answers in table below:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8