P U M $\therefore$ C

Combinatorics B

1. I have a 2 by 4 grid of squares; how many ways can I shade at least one of the squares so that no two shaded squares share an edge?

2. Amir enters Fine Hall and sees the number 2 written on a blackboard. Amir can perform the following operation: he flips a coin, and if it is heads, he replaces the number x on the blackboard with $3x+1$; otherwise, he replaces x with $\lfloor x/3 \rfloor$. If Amir performs this operation four times, let $\frac{m}{n}$ denote the expected number of times that he writes the digit 1 on the blackboard, where m, n are relatively prime positive integers. Find $m+n$.

3. Alien Connor starts at $(0,0)$ and walks around on the integer lattice. Specifically, he takes one step of length one in a uniformly random cardinal direction every minute, unless his previous four steps were all in the same direction in which case he randomly picks a new direction to step in. Every time he takes a step, he leaves toxic air on the lattice point he just left, and the toxic cloud remains there for 150 seconds. After taking 5 steps in total, the probability that he has not encountered his own toxic waste can be written as $\frac{a}{b}$ for relatively prime positive integers a, b. Find $a+b$.

4. Let $\oplus$ denote the xor binary operation. Define $x \star y = (x+y) - (x \oplus y)$. Compute

$$\sum_{k=1}^{63} (k \star 45).$$

(*Remark:* The xor operator works as follows: when considered in binary, the kth binary digit of $a \oplus b$ is 1 exactly when the kth binary digits of a and b are different. For example, $5 \oplus 12 = 0101_2 \oplus 1100_2 = 1001_2 = 9$.)

5. The integers from 1 to 25, inclusive, are randomly placed into a 5 by 5 grid such that in each row, the numbers are increasing from left to right. If the columns from left to right are numbered 1, 2, 3, 4, and 5, then the expected column number of the entry 23 can be written as $\frac{a}{b}$ where a and b are relatively prime positive integers. Find $a+b$.

6. A sequence of integers $a_1, a_2, \ldots, a_n$ is said to be *sub-Fibonacci* if $a_1 = a_2 = 1$ and $a_i \leq a_{i-1} + a_{i-2}$ for all $3 \leq i \leq n$. How many sub-Fibonacci sequences are there with 10 terms such that the last two terms are both 20?

7. There are n assassins numbered from 1 to n, and all assassins are initially alive. The assassins play a game in which they take turns in increasing order of number, with assassin 1 getting the first turn, then assassin 2, etc., with the order repeating after assassin n has gone; if an assassin is dead when their turn comes up, then their turn is skipped and it goes to the next assassin in line. On each assassin's turn, they can choose to either kill the assassin who would otherwise move next or to do nothing. Each assassin will kill on their turn unless the only option for guaranteeing their own survival is to do nothing. If there are 2023 assassins at the start of the game, after an entire round of turns in which no one kills, how many assassins must remain?

8. For a positive integer n, let P_n be the set of sequences of $2n$ elements, each 0 or 1, where there are exactly n 1's and n 0's. I choose a sequence uniformly at random from P_n. Then, I partition this sequence into maximal blocks of consecutive 0's and 1's. Define $f(n)$ to be the expected value of the sum of squares of the block lengths of this uniformly random sequence. What is the largest integer value that $f(n)$ can take on?

Name:

Team:

Write answers in table below:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8