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Geometry A Solutions

1. Define a *common chord* between two intersecting circles to be the line segment connecting their two intersection points. Let $\omega_1, \omega_2, \omega_3$ be three circles of radii 3, 5, and 7, respectively. Suppose they are arranged in such a way that the common chord of ω_1 and ω_2 is a diameter of ω_1, the common chord of ω_1 and ω_3 is a diameter of ω_1, and the common chord of ω_2 and ω_3 is a diameter of ω_2. Compute the square of the area of the triangle formed by the centers of the three circles.

   Proposed by Eric Shen

   Answer: $\boxed{96}$

   By Pythagoras, the distance between the centers of circles ω_i and ω_j with $j > i$ is $\sqrt{r_j^2 - r_i^2}$. We seek the area of a triangle with sidelengths $\sqrt{16}$, $\sqrt{24}$, and $\sqrt{40}$. But this is a right triangle whose area is $\frac{1}{2} \cdot \sqrt{16} \cdot \sqrt{24} = 4\sqrt{6}$, and our answer is $(4\sqrt{6})^2 = 96$.

2. Let $\triangle ABC$ be an isosceles triangle with $AB = AC = \sqrt{7}$ and $BC = 1$. Let G be the centroid of $\triangle ABC$. Given $j \in \{0, 1, 2\}$, let T_j denote the triangle obtained by rotating $\triangle ABC$ about G by $2\pi j/3$ radians. Let $\mathcal{P}$ denote the intersection of the interiors of triangles T_0, T_1, T_2. If K denotes the area of $\mathcal{P}$, then $K^2 = \frac{a}{b}$ for relatively prime positive integers a, b. Find $a + b$.

   Proposed by Sunay Joshi

   Answer: $\boxed{1843}$

   Construct the equilateral triangle $\triangle AXY$ with sidelength 3 such that BC is the middle third of the side XY (with B closer to X, WLOG). Note that T_0, T_1, T_2 lie within $\triangle AXY$; they are simply $\triangle ABC$ rotated. Let M, N lie on AY, AX such that $AM/MY = 1/2$ and $AN/NX = 1/2$. Let $P = AB \cap MX$, $Q = NY \cap MX$, and $R = AC \cap NY$. Then it is easy to see by symmetry that we see $K = 3 \cdot [PQRG]$. Since $PQRG$ has perpendicular diagonals, its area is given by $\frac{1}{2} \cdot QG \cdot PR$. To compute PR, note by mass points that $AP/PB = 3/2$, hence by similar triangles $PR = 3/5 \cdot BC = 3/5$. By mass points, we also have hat AQ is half the height of $\triangle AXY$, hence $QG = AG - AQ = \sqrt{3}/4$. Solving for K, we find $K = 3/2 \cdot \sqrt{3}/4 \cdot 3/5 = 9\sqrt{3}/40$. Squaring yields $K^2 = 243/1600$ and our answer is $243 + 1600 = 1843$.

3. Let $\triangle ABC$ be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. Let D, E, and F be the midpoints of AB, BC, and CA respectively. Imagine cutting $\triangle ABC$ out of paper and then folding $\triangle AFD$ up along FD, folding $\triangle BED$ up along DE, and folding $\triangle CEF$ up along EF until A, B, and C coincide at a point G. The volume of the tetrahedron formed by vertices D, E, F, and G can be expressed as $\frac{p\sqrt{q}}{r}$, where p, q, and r are positive integers, p and r are relatively prime, and q is square-free. Find $p + q + r$.

   Proposed by Atharva Pathak

   Answer: $\boxed{80}$

   Let H_1 be the foot of the perpendicular from A to DF and let H_2 be the foot of the perpendicular from E to DF. Note that a 13-14-15 triangle is a 5-12-13 triangle glued to a 9-12-15 triangle along the side of length 12. Because $\triangle ADF$ and $\triangle EFD$ are similar to $\triangle ABC$ scaled by a factor of $1/2$, we get that $AH_1 = EH_2 = 6$, $DH_1 = FH_2 = \frac{5}{2}$, and $H_1H_2 = 2$. Let θ be the dihedral angle between $\triangle GDF$ and $\triangle EDF$ in the tetrahedron. Because GE came from BE and CE in the original triangle, we have $GE = 7$. Now imagine projecting points G, H_1, H_2, and E onto a plane perpendicular to FD, such that G maps to G', H_1 and H_2 map to H', and E maps to E'. Since GE has a component of length $H_1H_2 = 2$ perpendicular to the plane, we get $G'E' = \sqrt{7^2 - 2^2} = \sqrt{45}$. Applying the law of cosines to $\triangle G'H'E'$ with the angle θ at H' gives $\cos\theta = \frac{3}{8}$. So the height of the tetrahedron, which is the distance from G' to $H'E'$, is $6\sin\theta = \frac{3\sqrt{55}}{4}$. Finally, the area of the base of the tetrahedron, i.e. $\triangle DEF$, is $\frac{1}{2}(7)(6) = 21$, so the volume is $\frac{1}{3}(21)\left(\frac{3\sqrt{55}}{4}\right) = \frac{21\sqrt{55}}{4}$, which gives a final answer of 80.

4. Let $\triangle ABC$ be a triangle with $AB = 4$, $BC = 6$, and $CA = 5$. Let the angle bisector of $\angle BAC$ intersect BC at the point D and the circumcircle of $\triangle ABC$ again at the point $M \neq A$. The perpendicular bisector of segment DM intersects the circle centered at M passing through B at two points, X and Y. Compute $AX \cdot AY$.

   Proposed by Eric Shen

   Answer: $\boxed{36}$

   Note that $AX = AY$ by symmetry and that $AX = AM$ by inversion about M. In a 4-5-6 triangle we have the following relation between the angles: $A = 2C$. Since AM subtends an angle of $\frac{A}{2} + C$ and since $\frac{A}{2} + C = A$, it follows that $AM = BC = 6$. Our answer is $6^2 = 36$.

5. Let $\triangle ABC$ have $AB = 15$, $AC = 20$, and $BC = 21$. Suppose ω is a circle passing through A that is tangent to segment BC. Let point $D \neq A$ be the second intersection of AB with ω, and let point $E \neq A$ be the second intersection of AC with ω. Suppose DE is parallel to BC. If $DE = \frac{a}{b}$, where a, b are relatively prime positive integers, find $a + b$.

   Proposed by Frank Lu

   Answer: $\boxed{361}$

   First, since DE is parallel to BC, we have that triangles ADE, ABC are similar. Furthermore, we have a homothety that sends triangle ADE to ABC. Notice that the image of this homothety also sends ω to the circumcircle of ABC. We thus need to determine the ratio of this homothety.

   To do this, let X be the tangency point of ω to BC. Draw line AX, and let M be the second intersection of line AX with the circumcircle. Then, we know from homothety that $\frac{AX}{AM} = \frac{DE}{BC}$; we just need to compute AX, AM. We furthermore note that ω is tangent to the circumcircle by homothety. From this configuration, we thus find that M is the midpoint of the minor arc BC, meaning that AX is the angle bisector of angle $\angle BAC$. First, to compute AX, we can employ Stewart's theorem. We know from the angle bisector theorem that $\frac{BX}{CX} = \frac{3}{4}$, meaning that $BX = 9$ and $CX = 12$. Therefore, we have by Stewart's theorem that $AX^2 \cdot 21 + 9 \cdot 12 \cdot 21 = 15^2 \cdot 12 + 20^2 \cdot 9$, or that $21AX^2 = 5^2 \cdot 3 \cdot (3^2 \cdot 4 + 4^2 \cdot 3) - 9 \cdot 12 \cdot 21 = 5^2 \cdot 3 \cdot 4 \cdot 21 - 9 \cdot 12 \cdot 21$, or that $AX^2 = 300 - 108 = 192$, so $AX = 8\sqrt{3}$.

   From here, we compute AM. The method that we use to compute this is Ptolemy's theorem and Law of Cosines chasing. First, consider BM, CM. Note that $\angle BMC = 180 - \angle BAC$, and so $\cos\angle BAC = -\cos\angle BMC$. But now by Law of Cosines, we know that $\cos\angle BAC = \frac{15^2+20^2-21^2}{2\cdot 15\cdot 20} = \frac{23}{75}$. Therefore, we have that $BC^2 = BM^2(2-2\cos\angle BMC) = BM^2\frac{196}{75}$, meaning that $BM = CM = \frac{15\sqrt{3}}{2}$. Finally, by Ptolemy's Theorem, we have that $AM \cdot BC = BM(AB + AC)$, or that $AM = \frac{35}{21}\frac{15\sqrt{3}}{2} = \frac{25\sqrt{3}}{2}$.

   It follows that $DE = \frac{AX}{AM}BC = \frac{336}{25}$, so our answer is 361.

6. Let $\triangle ABC$ have $AB = 14$, $BC = 30$, $AC = 40$ and $\triangle AB'C'$ with $AB' = 7\sqrt{6}$, $B'C' = 15\sqrt{6}$, $AC' = 20\sqrt{6}$ such that $\angle BAB' = \frac{5\pi}{12}$. The lines BB' and CC' intersect at point D. Let O be the circumcenter of $\triangle BCD$, and let O' be the circumcenter of $\triangle B'C'D$. Then the length of segment OO' can be expressed as $\frac{a+b\sqrt{c}}{d}$, where a, b, c, and d are positive integers such that a and d are relatively prime, and c is not divisible by the square of any prime. Find $a+b+c+d$.

   Proposed by Adam Huang

Answer: $\boxed{55}$

Note that $\triangle ABC$ and $\triangle AB'C'$ are spirally similar with center of spiral similarity given by A and angle $\frac{5\pi}{12}$ and dilation factor $\frac{\sqrt{6}}{2}$. By properties of spiral similarity, we have that $D := BB' \cap CC'$ lies on circumcircles (ABC) and $(AB'C')$. Therefore AO is the circumradius of $\triangle ABC$, and $AO' = \frac{\sqrt{6}}{2}AO$ by similarity, with $\angle OAO' = \frac{5\pi}{12}$. To compute $R := AO$, note by Heron that the area of $\triangle ABC$ is $K = 168$, so that $\frac{abc}{4R} = K \implies R = \frac{abc}{4K} = \frac{14\cdot30\cdot40}{4\cdot168} = 25$. By Law of Cosines, we have $(OO')^2 = 25^2\cdot(1^2+(\frac{\sqrt{6}}{2})^2 - 2\cdot1\cdot\frac{\sqrt{6}}{2}\cdot\cos\frac{5\pi}{12})$, so that $OO' = \frac{25+25\sqrt{3}}{2}$, which yields an answer of $a+b+c+d = 25+25+3+2 = 55$.

7. Let $\triangle ABC$ be a triangle with $\angle BAC = 90°$, $\angle ABC = 60°$, and $\angle BCA = 30°$ and $BC = 4$. Let the incircle of $\triangle ABC$ meet sides BC, CA, AB at points A_0, B_0, C_0, respectively. Let $\omega_A, \omega_B, \omega_C$ denote the circumcircles of triangles $\triangle B_0IC_0, \triangle C_0IA_0, \triangle A_0IB_0$, respectively. We construct triangle T_A as follows: let A_0B_0 meet ω_B for the second time at $A_1 \neq A_0$, let A_0C_0 meet ω_C for the second time at $A_2 \neq A_0$, and let T_A denote the triangle $\triangle A_0A_1A_2$. Construct triangles T_B, T_C similarly. If the sum of the areas of triangles T_A, T_B, T_C equals $\sqrt{m} - n$ for positive integers m, n, find $m+n$.

Proposed by Sunay Joshi

Answer: $\boxed{15}$

We begin by computing the inradius r. The sides of $\triangle ABC$ are clearly $a = 4$, $b = 2\sqrt{3}$, and $c = 2$, so that the semiperimeter is $s = 3+\sqrt{3}$. The area is $2\sqrt{3}$. Since $rs = [ABC]$, we have $r = \frac{2\sqrt{3}}{3+\sqrt{3}}$.

We now compute the area of T_B. For convenience we relabel A_0, B_0, C_0 by D, E, F. Let r_A, r_B, r_C denote the radii of $\omega_A, \omega_B, \omega_C$, respectively. Note that by trigonometry, $\frac{r}{2r_A} = \sin\frac{A}{2}$ and $EF = 2\cos\frac{A}{2}\cdot r$, with similar equalities holding for cyclic permutations of the indices. Let us take $B_1 = FE \cap \omega_C$ WLOG. The key observation is the following spiral similarity: $\triangle DFB_1 \sim \triangle DII_C$, where I_C is the center of ω_C. Therefore $\frac{DF}{FB_1} = \frac{DI}{II_C}$, so that $FB_1 = \frac{II_C}{DI}\cdot DF$, so that

$$FB_1 = \frac{r_C}{r}\cdot 2r_B\sin B \tag{1}$$
$$= \frac{1}{2\sin\frac{C}{2}}\cdot 2\cos\frac{B}{2}\cdot r \tag{2}$$

Thus

$$EB_1 = FB_1 - EF \tag{3}$$
$$= \frac{1}{2\sin\frac{C}{2}}\cdot 2\cos\frac{B}{2}\cdot r - 2\cos\frac{A}{2}\cdot r \tag{4}$$
$$= r\cdot\frac{1}{\sin\frac{C}{2}}\cdot(\cos\frac{B}{2} - 2\cos\frac{A}{2}\sin\frac{C}{2}) \tag{5}$$
$$= r\cdot\frac{1}{\sin\frac{C}{2}}\cdot\sin|\frac{A-C}{2}| \tag{6}$$

By symmetry, $EB_2 = r\cdot\frac{1}{\sin\frac{A}{2}}\cdot\sin|\frac{A-C}{2}|$. Since $\angle B_1EB_2 = 90 - \frac{B}{2}$, the sine area formula applied to T_B yields

$$[T_B] = \frac{1}{2}r^2\cdot\cos\frac{B}{2}\cdot\frac{1}{\sin\frac{A}{2}\sin\frac{C}{2}}\cdot\sin^2\frac{A-C}{2} \tag{7}$$

which may be simplified via half-angle and product-to-sum as follows:

$$[T_B] = \frac{1}{4}r^2\frac{\sin B}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\sin^2\frac{A-C}{2} \tag{8}$$

$$= \frac{1}{8}r^2\frac{\sin B}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}(1-\cos(A-C)) \tag{9}$$

$$= \frac{1}{8}r^2\frac{1}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}(\sin B-\cos(A-C)\sin B) \tag{10}$$

$$= \frac{1}{8}r^2\frac{1}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}(\sin B-\frac{1}{2}(\sin 2A+\sin 2C)) \tag{11}$$

Summing cyclically, we find

$$[T_A]+[T_B]+[T_C] = \frac{1}{8}r^2\frac{1}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}(\sin A+\sin B+\sin C-\sin 2A-\sin 2B-\sin 2C) \tag{12}$$

Recall $A=90, B=60, C=30$, so that $\sin A=1$, $\sin 2A=0$, $\sin\frac{A}{2}=\frac{1}{\sqrt{2}}$, $\sin B=\frac{\sqrt{3}}{2}$, $\sin 2B=\frac{\sqrt{3}}{2}$, $\sin\frac{B}{2}=\frac{1}{2}$, $\sin C=\frac{1}{2}$, $\sin 2C=\frac{\sqrt{3}}{2}$, $\sin\frac{C}{2}=\frac{\sqrt{3}-1}{2\sqrt{2}}$. Plugging in, the sum becomes

$$\frac{1}{8}r^2\frac{1}{\frac{1}{\sqrt{2}}\cdot\frac{1}{2}\cdot\frac{\sqrt{3}-1}{2\sqrt{2}}}\cdot(1+\frac{\sqrt{3}}{2}+\frac{1}{2}-0-\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}) \tag{13}$$

$$= (\frac{2\sqrt{3}}{3+\sqrt{3}})^2\frac{1}{\sqrt{3}-1}\frac{3-\sqrt{3}}{2} \tag{14}$$

$$= 2\sqrt{3}-3 \tag{15}$$

Therefore the sum is $\sqrt{12}-3$ with $m=12$ and $n=3$, yielding an answer of $12+3=15$.

8. Similar to the last 6 problems, let $\triangle ABC$ be a triangle with $AB=4$ and $AC=\frac{7}{2}$. Let ω denote the A-excircle of $\triangle ABC$. Let ω touch lines AB, AC at the points D, E, respectively. Let Ω denote the circumcircle of $\triangle ADE$. Consider the line ℓ parallel to BC such that ℓ is tangent to ω at a point F and such that ℓ does not intersect Ω. Let ℓ intersect lines AB, AC at the points X, Y, respectively, with $XY=18$ and $AX=16$. Let the perpendicular bisector of XY meet the circumcircle of $\triangle AXY$ at P, Q, where the distance from P to F is smaller than the distance from Q to F. Let ray $\overrightarrow{PF}$ meet Ω for the first time at the point Z. If $PZ^2=\frac{m}{n}$ for relatively prime positive integers m, n, find $m+n$.

   Proposed by Sunay Joshi

   Answer: $\boxed{1159}$

   Below, we let (XYZ) denote the circumcircle of the triangle $\triangle XYZ$.

   We restate the problem (changing the names of points) as follows: let $\triangle ABC$ be a triangle with sidelengths $a=9$, $b=7$, and $c=8$. Let M denote the midpoint of the minor arc BC in (ABC). Let D denote the point of tangency between the incircle ω of $\triangle ABC$ and BC. Let $\vec{MD}$ intersect ω for the first time at X. We seek MX^2. Note that our problem is simply this, scaled up by a factor of 2. Therefore at the end, we multiply the value of MX^2 we obtain by a factor of 4.

   The first key observation is that P, X, M are collinear, where $P \neq A$ is the second intersection of ω and (ABC). The next key observation is the spiral similarity $\triangle AXM \sim \triangle AI_AO$, where

I_A denotes the center of (API). This yields $\frac{MX}{MA} = \frac{OI_A}{OA}$, which when rearranged implies $MX = MA \cdot \frac{OI_A}{R}$.

To find MA, we may apply Ptolemy's theorem to the cyclic quadrilateral $ABMC$: $MA \cdot a = (b+c) \cdot BM$, which implies $MA = MB \cdot \frac{b+c}{a} = 2R \sin \frac{A}{2} \cdot \frac{b+c}{a}$.

To find OI_A, we apply the power of a point to I_A and the circle (ABC). The power of I_A is given by $AI_A \cdot I_AM$, or equivalently $\frac{1}{2}AI \cdot (AM - \frac{1}{2}AI)$. By the definition of power, this also equals $R^2 - OI_A^2$.

By trigonometry, we have $AI = \frac{r}{\sin \frac{A}{2}}$. By the Law of Sines in (ABC), we have $MA = 2R\sin(\frac{A}{2} + B)$. By the Law of Cosines, we find $\cos A = 2/7$, $\sin \frac{A}{2} = \sqrt{5/14}$, $\cos \frac{A}{2} = 3/\sqrt{14}$, $\cos B = 2/3$, $\sin B = \sqrt{5}/3$. Also, we find $R = \frac{21}{2\sqrt{5}}$ and $r = \sqrt{5}$. Therefore $AI = \frac{\sqrt{5}}{\sqrt{5/14}} = \sqrt{14}$ and $\frac{1}{2}AI = \frac{\sqrt{14}}{2}$. Also, by the sine addition formula, $\sin(\frac{A}{2} + B) = \sin \frac{A}{2} \cos B + \sin B \cos \frac{A}{2} = \frac{5}{3}\sqrt{\frac{5}{14}}$. Next, $AM = 2 \cdot \frac{21}{2\sqrt{5}} \cdot \frac{5}{3}\frac{\sqrt{5}}{\sqrt{14}} = \frac{5\sqrt{14}}{2}$. Hence $AM - \frac{1}{2}AI = 2\sqrt{14}$. Thus the power of I_A equals $\frac{1}{2}\sqrt{14} \cdot 2\sqrt{14} = 14$. Equating the two expressions for power, we have $OI_A^2 = R^2 - 14 = \frac{21^2}{20} - 14 = \frac{161}{20}$. Next, $MA = 2R\sin\frac{A}{2} \cdot \frac{b+c}{a}$, which reduces to $\frac{5\sqrt{14}}{2}$, so that $MA^2 = \frac{175}{2}$. Finally, note $R^2 = (\frac{21}{2\sqrt{5}})^2 = \frac{441}{20}$.

Putting everything together, we find $MX^2 = MA^2 \cdot \frac{OI_A^2}{R^2} = \frac{175}{2} \cdot \frac{\frac{161}{20}}{\frac{441}{20}} = \frac{575}{18}$. Recall that we must scale up by a factor of 4. Therefore the true value of MX^2 is $MX^2 = 4 \cdot \frac{575}{18} = \frac{1150}{9}$, so that our answer is $1150 + 9 = 1159$.