Geometry B Solutions

1. A triangle $\triangle ABC$ is situated on the plane and a point E is given on segment AC. Let D be a point in the plane such that lines AD and BE are parallel. Suppose that $\angle EBC = 25°, \angle BCA = 32°$, and $\angle CAB = 60°$. Find the smallest possible value of $\angle DAB$ in degrees.

   Proposed by Frank Lu

   Answer: $\boxed{63}$

   First, using the angles that we are given, we can compute that $\angle BEC = 180° - 57° = 123°$. From here, we have two cases, depending on the positioning of D. In the first case, we have that $\angle DAC = \angle BEC$ (rays BE and AD point in opposite directions). In this case, we have $\angle DAC = 123°$, meaning that $\angle DAB = \angle DAC - \angle DAB = 63°$.

   In the second case, $\angle DAC = 180° - \angle BEC = 57°$, with rays BE, AD pointing the same way. In this case, we have that $\angle DAB = \angle DAC + \angle CAB = 117°$. Notice that of these cases, the smallest value is $63°$, which gives us our answer.

2. Three spheres are all externally tangent to a plane and to each other. Suppose that the radii of these spheres are $6, 8$, and, 10. The tangency points of these spheres with the plane form the vertices of a triangle. Determine the largest integer that is smaller than the perimeter of this triangle.

   Proposed by Frank Lu

   Answer: $\boxed{47}$

   Suppose that the centers of the spheres are O_1, O_2, O_3, with the respective tangency points to the planes being A_1, A_2, A_3, so that the sphere centered at O_1 has radius 6, the sphere centered at O_2 has radius 8, and the sphere centered at O_3 has radius 10. Our goal will to be to compute A_1A_2, A_2A_3, A_3A_1. We go through the computation for A_1A_2, and the computation for the other two sides follows similarly.

   Notice that, considering the plane containing A_1, A_2, O_1, O_2, draw the line perpendicular to O_2A_2 that goes through O_1. Say that this line intersects O_2A_2 at X_1. Then, $O_1X_1 = A_1A_2$, and we know that $O_2X_1 = O_2A_2 - O_1A_1 = 1$. Therefore, it follows that, by the Pythagorean theorem, as $O_1O_2 = 14$, we have that the length of A_1A_2 is $\sqrt{14^2 - 2^2}$.

   Similarly, we have that $A_2A_3 = \sqrt{18^2 - 2^2}$ and $A_3A_1 = \sqrt{16^2 - 4^2}$. Notice that the sum of these values is less than $14 + 18 + 16 = 48$. We argue that this sum is larger than 47. To see this, observe that $(n - 1/3)^2 = n^2 - 2n/3 + 1/9 < n^2 - 4$, for $n \geq 14$. Therefore, it follows that $\sqrt{14^2 - 2^2} > 14 - 1/3$, and similarly for the other two sides. Summing up these inequalities yields that the perimeter is larger than 47. This is our desired floor.

3. Circle Γ is centered at $(0, 0)$ in the plane with radius $2022\sqrt{3}$. Circle Ω is centered on the x-axis, passes through the point $A = (6066, 0)$, and intersects Γ orthogonally at the point $P = (x, y)$ with $y > 0$. If the length of the minor arc AP on Ω can be expressed as $\frac{m\pi}{n}$ for relatively prime positive integers m, n, find $m + n$.

   (Two circles are said to intersect *orthogonally* at a point P if the tangent lines at P form a right angle.)

   Proposed by Sunay Joshi

   Answer: $\boxed{1349}$

   Let $O = (0, 0)$. Let $R = 2022\sqrt{3}$ denote the radius of Γ, so that $OA = R\sqrt{3}$. Let r denote the radius of Ω. Let Q denote the center of Ω. Since OPQ is a right triangle, $PQ = \sqrt{r^2 + R^2}$.

Since $OA = OQ + QA$, we have

$$\sqrt{R^2 + r^2} + r = R\sqrt{3}$$

Solving, we find that $r = R/\sqrt{3} = 2022$. Therefore $\angle OQP = 60^\circ$ and the minor arc AP corresponds to an interior angle of 120°. It follows that the desired arclength is given as $\frac{1}{3} \cdot 2\pi r = \frac{4044\pi}{3} = \frac{1348\pi}{1}$, and our answer is $1348 + 1 = 1349$.

4. An ellipse has foci A and B and has the property that there is some point C on the ellipse such that the area of the circle passing through A, B, and, C is equal to the area of the ellipse. Let e be the largest possible eccentricity of the ellipse. One may write e^2 as $\frac{a+\sqrt{b}}{c}$, where a, b, and c are integers such that a and c are relatively prime, and b is not divisible by the square of any prime. Find $a^2 + b^2 + c^2$.

   Proposed by Daniel Carter

   Answer: $\boxed{30}$

   Consider the ellipse with largest possible eccentricity that has this property. The smallest possible area of the circle is when the center of the circle is the center of the ellipse. Let O be the center of the ellipse. Then $\pi(OA)^2 = \pi Rr$, where R, r are the semi-major and semi-minor axes. We have $OA/R = e$, so then $(OA)e = r$. Noting that $r^2 = R^2 - (OA)^2$, this means $e^2 = (1/e^2 - 1)$, or $e^2 = \frac{-1+\sqrt{5}}{2}$. So the answer is $(-1)^2 + 5^2 + 2^2 = 30$.

5. Daeun draws a unit circle centered at the origin and inscribes within it a regular hexagon $ABCDEF$. Then Dylan chooses a point P within the circle of radius 2 centered at the origin. Let M be the maximum possible value of $|PA| \cdot |PB| \cdot |PC| \cdot |PD| \cdot |PE| \cdot |PF|$, and let N be the number of possible points P for which this maximal value is obtained. Find $M + N^2$.

   Proposed by Dylan Epstein-Gross

   Answer: $\boxed{101}$

   Using roots of unity, the product of lengths is

$$|z-1||z-a||z-a^2| \cdots |z-a^5| = |z^6 - 1|$$

   This is maximized when $z^6 = -64$, which has six solutions with $M = 65$. Thus the answer is $65 + 6^2 = 101$.

6. Let $\triangle ABC$ be an equilateral triangle. Points D, E, F are drawn on sides AB, BC, and CA respectively such that $[ADF] = [BED] + [CEF]$ and $\triangle ADF \sim \triangle BED \sim \triangle CEF$. The ratio $\frac{[ABC]}{[DEF]}$ can be expressed as $\frac{a+b\sqrt{c}}{d}$, where a, b, c, and d are positive integers such that a and d are relatively prime, and c is not divisible by the square of any prime. Find $a + b + c + d$.

   (Here $[\mathcal{P}]$ denotes the area of polygon $\mathcal{P}$.)

   Proposed by Adam Huang

   Answer: $\boxed{17}$

   Assume WLOG that $\triangle ABC$ has sidelength 1. The similarity condition implies $DE^2 + EF^2 = DF^2$, hence $\angle DEF = 90$. Angle chasing also yields $\angle BED = 45$, so that $\triangle BED, \triangle CEF$ are 60-45-75 triangles and $\triangle DEF$ is a 30-60-90 right triangle. By the Law of Sines applied to $\triangle BED$ and $\triangle CEF$, the lengths $z = DE$ and $x = BE$ satisfy $\frac{z}{\sin 60} = \frac{x}{\sin 75}$ and $\frac{z\sqrt{3}}{\sin 60} = \frac{1-x}{\sin 75}$. Solving, we find $x = \frac{\sqrt{3}-1}{2}$ and $z = \frac{2\sqrt{6}-3\sqrt{2}}{2}$. Thus $[DEF] = \frac{z^2\sqrt{3}}{2}$ and the ratio is $[ABC]/[DEF] = \frac{\sqrt{3}/4}{z^2\sqrt{3}/2}$, which reduces to $\frac{7+4\sqrt{3}}{3}$. Our answer is $7 + 4 + 3 + 3 = 17$.

7. Let $\triangle ABC$ be a triangle with $AB = 5$, $BC = 8$, and, $CA = 7$. Let the center of the A-excircle be O, and let the A-excircle touch lines BC, CA, and, AB at points X, Y, and, Z, respectively. Let h_1, h_2, and, h_3 denote the distances from O to lines XY, YZ, and, ZX, respectively. If $h_1^2 + h_2^2 + h_3^2$ can be written as $\frac{m}{n}$ for relatively prime positive integers m, n, find $m + n$.

   Proposed by Sunay Joshi

   Answer: $\boxed{2189}$

   Let a, b, c denote the lengths of sides BC, CA, AB, and let r_A denote the radius of the A-excircle. We claim that $h_1 = \frac{r_A^2 \sin\frac{C}{2}}{s-b}$, $h_2 = \frac{r_A^2 \sin\frac{B}{2}}{s-c}$, and $h_3 = \frac{r_A^2 \cos\frac{A}{2}}{s}$. We begin with h_1. Computing the area of $\triangle OXY$ in two ways, we find $\frac{1}{2}h_1 \cdot XY = \frac{1}{2}r_A^2 \sin XOY$. Since $XY = 2(s-b)\cos\frac{C}{2}$ and $\angle XOY = C$, solving the equation for h_1 yields the desired formula. By symmetry, this implies the expression for h_3. For h_2, we compute the area of $\triangle YOZ$ in two ways to find $\frac{1}{2}h_2 \cdot YZ = \frac{1}{2}r_A^2 \sin YOZ$. Since $\angle YOZ = \pi - A$ and $YZ = 2s\sin\frac{A}{2}$, solving the equation for h_2 yields the desired formula.

   Having established the above, we now compute each of h_1, h_2, h_3. By the Law of Cosines, $\cos A = \frac{1}{7}$, $\cos B = \frac{1}{2}$, and $\cos C = \frac{11}{14}$. By the half-angle formulae, it follows that $\cos\frac{A}{2} = \frac{2}{\sqrt{7}}$, $\sin\frac{B}{2} = \frac{1}{2}$, and $\sin\frac{C}{2} = \frac{\sqrt{3}}{2\sqrt{7}}$. Next, since $r_A(s-a) = K$, Heron's formula implies that $r_A^2 = \frac{s(s-b)(s-c)}{s-a} = 75$. Putting everything together, we find that

$$h_1^2 + h_2^2 + h_3^2 = 75^2 \cdot \left[\left(\frac{2/\sqrt{7}}{10}\right)^2 + \left(\frac{\sqrt{3}/(2\sqrt{7})}{3}\right)^2 + \left(\frac{1/2}{5}\right)^2 \right] = \frac{2175}{14}$$

   This gives an answer of $m + n = 2189$.

8. Triangle $\triangle ABC$ has sidelengths $AB = 10, AC = 14$, and, $BC = 16$. Circle ω_1 is tangent to rays $\overrightarrow{AB}, \overrightarrow{AC}$ and passes through B. Circle ω_2 is tangent to rays $\overrightarrow{AB}, \overrightarrow{AC}$ and passes through C. Let ω_1, ω_2 intersect at points X, Y. The square of the perimeter of triangle $\triangle AXY$ is equal to $\frac{a+b\sqrt{c}}{d}$, where a, b, c, and, d are positive integers such that a and d are relatively prime, and c is not divisible by the square of any prime. Find $a + b + c + d$.

   Proposed by Frank Lu

   Answer: $\boxed{6272}$

   Draw the angle bisector of BAC, which we denote as ℓ. Notice that if O_1 is the center of ω_1 and O_2 is the center of ω_2, then we have that O_1, O_2 lie on this angle bisector. It follows that this angle bisector must be the perpendicular bisector of XY, since XY is the radical axis of these two circles.

   We first compute AP. To do this, consider the following operation: first, reflect the diagram about the angle bisector, then perform an inversion about A of radius $\sqrt{10 \cdot 14}$. (This latter inversion is referred to sometimes as a root bc inversion). Notice that this operation sends the circle ω_1 to ω_2, and sends X to Y. Furthermore, since ℓ is the perpendicular bisector of XY, we have that $AY = AX = \frac{140}{AX}$, meaning that $AX = \sqrt{140}$. To find XP, we need to now find AP. But this can be done by considering the triangle O_1XO_2.

   We compute the side lengths of this triangle. First, we know that XO_1 is the radius of ω_1. We can then compute XO_1 by considering the incircle: if the incircle has radius r, and l is the length of the tangent from A to the incircle, we know that $\frac{l}{r} = \frac{AB}{BO_1} = \frac{AB}{XO_1}$. But if s is the semi-perimeter of ABC, we know that $l = s - BC = 4$, and r is equal to $\sqrt{\frac{(s-AB)(s-BC)(s-AC)}{s}} = \sqrt{\frac{10\cdot 6\cdot 4}{20}} = 2\sqrt{3}$. Therefore, we see that $XO_1 = 5\sqrt{3}$. Similarly, $XO_2 = 7\sqrt{3}$. Finally, we can then compute that $O_1O_2 = AO_2 - AO_1$, which using the Pythagorean theorem is $7\sqrt{7} - 5\sqrt{7} = 2\sqrt{7}$.

Therefore, we compute that $PO_2^2 - PO_1^2 = 72$. Since this is larger than $2\sqrt{7}$, we see that our triangle is obtuse. Thus, we have that $O_1P + O_2P = \frac{36\sqrt{7}}{7}$, and $OP_2 - OP_1 = 2\sqrt{7}$, which gives us that $O_1P = \frac{11\sqrt{7}}{7}$, and so therefore $AP = AO_1 + O_1P = \frac{24\sqrt{7}}{7}$.

From here, we compute that $XP^2 = 140 - \frac{576}{7} = \frac{404}{7}$, or that $XP = 2\sqrt{\frac{101}{7}}$. Therefore, the perimeter of our triangle is equal to $4\sqrt{\frac{101}{7}} + 4\sqrt{35} = \frac{\sqrt{1616}+\sqrt{3920}}{\sqrt{7}}$. Therefore the square of the perimeter is $\frac{5536+224\sqrt{505}}{7}$, so our answer is $a + b + c + d = 5536 + 224 + 505 + 7 = 6272$.