Geometry B

1. Rectangle $ABCD$ has $AB = 24$ and $BC = 7$. Let d be the distance between the centers of the incircles of $\triangle ABC$ and $\triangle CDA$. Find d^2.

2. The area of the largest square that can be inscribed in a regular hexagon with sidelength 1 can be expressed as $a - b\sqrt{c}$ where c is not divisible by the square of any prime. Find $a+b+c$.

3. Define a *common chord* between two intersecting circles to be the line segment connecting their two intersection points. Let $\omega_1, \omega_2, \omega_3$ be three circles of radii 3, 5, and 7, respectively. Suppose they are arranged in such a way that the common chord of ω_1 and ω_2 is a diameter of ω_1, the common chord of ω_1 and ω_3 is a diameter of ω_1, and the common chord of ω_2 and ω_3 is a diameter of ω_2. Compute the square of the area of the triangle formed by the centers of the three circles.

4. Let $\triangle ABC$ be an isosceles triangle with $AB = AC = \sqrt{7}$ and $BC = 1$. Let G be the centroid of $\triangle ABC$. Given $j \in \{0, 1, 2\}$, let T_j denote the triangle obtained by rotating $\triangle ABC$ about G by $2\pi j/3$ radians. Let $\mathcal{P}$ denote the intersection of the interiors of triangles T_0, T_1, T_2. If K denotes the area of $\mathcal{P}$, then $K^2 = \frac{a}{b}$ for relatively prime positive integers a, b. Find $a + b$.

5. Let $\triangle ABC$ be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. Let D, E, and F be the midpoints of AB, BC, and CA respectively. Imagine cutting $\triangle ABC$ out of paper and then folding $\triangle AFD$ up along FD, folding $\triangle BED$ up along DE, and folding $\triangle CEF$ up along EF until A, B, and C coincide at a point G. The volume of the tetrahedron formed by vertices D, E, F, and G can be expressed as $\frac{p\sqrt{q}}{r}$, where p, q, and r are positive integers, p and r are relatively prime, and q is square-free. Find $p + q + r$.

6. Let $\triangle ABC$ be a triangle with $AB = 4$, $BC = 6$, and $CA = 5$. Let the angle bisector of $\angle BAC$ intersect BC at the point D and the circumcircle of $\triangle ABC$ again at the point $M \neq A$. The perpendicular bisector of segment DM intersects the circle centered at M passing through B at two points, X and Y. Compute $AX \cdot AY$.

7. Let $\triangle ABC$ have $AB = 15$, $AC = 20$, and $BC = 21$. Suppose ω is a circle passing through A that is tangent to segment BC. Let point $D \neq A$ be the second intersection of AB with ω, and let point $E \neq A$ be the second intersection of AC with ω. Suppose DE is parallel to BC. If $DE = \frac{a}{b}$, where a, b are relatively prime positive integers, find $a + b$.

8. Let $\triangle ABC$ have $AB = 14$, $BC = 30$, $AC = 40$ and $\triangle AB'C'$ with $AB' = 7\sqrt{6}$, $B'C' = 15\sqrt{6}$, $AC' = 20\sqrt{6}$ such that $\angle BAB' = \frac{5\pi}{12}$. The lines BB' and CC' intersect at point D. Let O be the circumcenter of $\triangle BCD$, and let O' be the circumcenter of $\triangle B'C'D$. Then the length of segment OO' can be expressed as $\frac{a+b\sqrt{c}}{d}$, where $a, b, c,$ and d are positive integers such that a and d are relatively prime, and c is not divisible by the square of any prime. Find $a+b+c+d$.

(Write answers on next page.)

PUMaC

Name:

Team:

Write answers in table below:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8