P U M . . C

Individual Finals A

1. Let $p > 3$ be a prime and $k \geq 0$ an integer. Find the multiplicity of $X - 1$ in the factorization of
$$f(X) = X^{3p^k-1} + X^{3p^k-2} + \cdots + X + 1$$
modulo p; in other words, find the unique non-negative integer r such that $(X-1)^r$ divides $f(X)$ modulo p, but $(X-1)^{r+1}$ does not divide $f(X)$ modulo p.

2. On an infinite triangular lattice, there is a single atom at a lattice point. We allow for four operations as illustrated in Figure 1. In words, one could take an existing atom, split it into three atoms, and place them at adjacent lattice points in one of the two displayed fashions (a "split"). One could also reverse the process, i.e. taking three existing atoms in the displayed configurations, and merge them into a single atom at the center (a "merge").

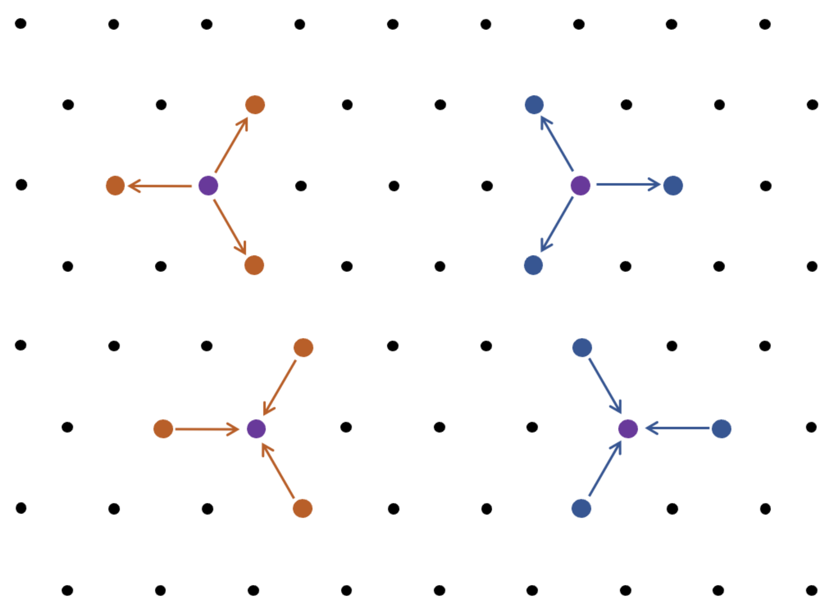

Figure 1: The four possible operations on an atom.

Assume that, after finitely many operations, there is again only a single atom remaining on the lattice. Show that this is possible if and only if the final atom is contained in the sublattice implied by Figure 2.

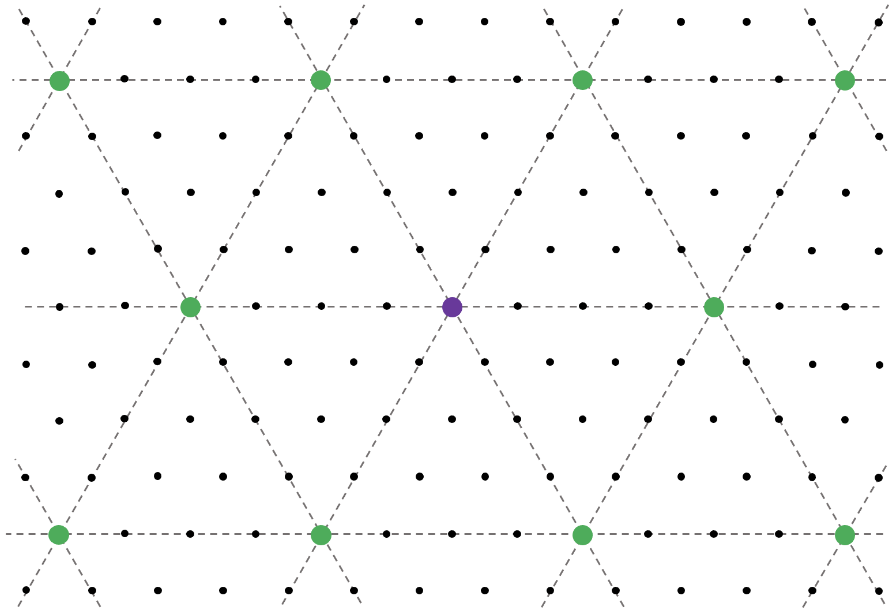

Figure 2: The possible positions for the final atom is the green sublattice. The position of the original atom is marked in purple.

3. Let $f(X)$ be a monic irreducible polynomial over $\mathbb{Z}$; therefore, by Gauss's Lemma, f is also irreducible over $\mathbb{Q}$ (you may assume this). Moreover, assume $f(X) \mid f(X^2+n)$ where n is an integer such that $n \notin \{-1,0,1\}$. Show that $n^2 \nmid f(0)$.