P U M ∴ C

Individual Finals B

1. Let a, b, c, d be real numbers for which $a^2 + b^2 + c^2 + d^2 = 1$. Show the following inequality:

$$a^2 + b^2 - c^2 - d^2 \leq \sqrt{2 + 4(ac + bd)}.$$

2. Given a triangle $\triangle ABC$, construct squares $BAQP$ and $ACRS$ outside the triangle ABC (with vertices in that listed in counterclockwise order). Show that the line from A perpendicular to BC passes through the midpoint of the segment QS.

3. Anna and Bob play the following game. In the beginning, Bob writes down the numbers $1, 2, \ldots, 2022$ on a piece of paper, such that half of the numbers are on the left and half on the right. Furthermore, we assume that the 1011 numbers on both sides are written in some order.

   After Bob does this, Anna has the opportunity to swap the positions of the two numbers lying on different sides of the paper if they have different parity. Anna wins if, after finitely many moves, all odd numbers end up on the left, in increasing order, and all even ones end up on the right, in increasing order. Can Bob write down a arrangement of numbers for which Anna cannot win?

   For example, Bob could write down numbers in the following way:

   $4, 2, 5, 7, 9, \ldots, 2021$ $\qquad\qquad$ $3, 1, 6, 8, 10, \ldots, 2022$

   Then Anna could swap the numbers $1, 4$ and then swap $2, 3$ to win. However, if Anna swapped the pairs $3, 4$ and $1, 2$, the resulting numbers on the left and on the right would not be in increasing order, and hence Anna would not win.