P U M ∴ C

Individual Finals B

1. For a binary string S (i.e. a string of 0's and 1's) that contains at least one 0, we produce a binary string $f(S)$ as follows:

   - If the substring 110 occurs in S, replace each instance of 110 with 01 to produce $f(S)$;
   - Otherwise, replace the leftmost occurrence of 0 in S by 1 to produce $f(S)$.

   Given binary string S of length n, we define the *lifetime* of S to be the number of times f can be applied to S until the resulting string contains no more 0's. For example,

   $$111000 \to 10100 \to 11100 \to 1010 \to 1110 \to 101 \to 111$$

   so the lifetime of 111000 is 6. For a given $n \geq 2$, which binary string(s) of length n have the longest lifetime? As usual, you need to rigorously justify your answers.

2. Let f be a polynomial with degree at most $n-1$. Show that

   $$\sum_{k=0}^{n} \binom{n}{k} (-1)^k f(k) = 0.$$

3. Let $p > 3$ be a prime and $k \geq 0$ an integer. Find the multiplicity of $X - 1$ in the factorization of

   $$f(X) = X^{3p^k-1} + X^{3p^k-2} + \cdots + X + 1$$

   modulo p; in other words, find the unique non-negative integer r such that $(X-1)^r$ divides $f(X)$ modulo p, but $(X-1)^{r+1}$ does not divide $f(X)$ modulo p.