P U M A C

PUMAC Solutions and Rubric-B

1 B1

Problem: Let a positive integer n have at least four positive divisors. Let the least four positive divisors be $1 = d1 < d2 < d3 < d4$. Find, with proof, all solutions to $n^2 = d_1^3 + d_2^3 + d_4^3$.

Proposed by Jackson Blitz.

Solution: Let $d2 = p$ be prime. Then taking the equation $\mod p$ we get: since $p|n$ we have $-1 \equiv d_4^3 (\mod n)$ So then d_4 must not have a divisor of p,so $d_4 = q$ or $d_4 = q^2$ where q prime and $q > p$. Taking $\mod q^2$ we get $q^2|p^3 + 1$. Since $q > p$ unless $p = 2$ and$q = 3$ we must have $q^2|p^2 - p + 1$, which is impossible as $q > p$. Considering when $p = 2$ and $q = 3$, then since it must be that $d_3 = 3 = q$ then $d_4 = 9$. So the equation is $n^2 = 1 + 2^3 + 3^6 = 3^2(1 + 3^4)$ which is not a square. So no solutions exist.

Rubric:

- Taking equation $\mod p$ (1 point).
- Concluding that $d_4 = q$ or $d_4 = q^2$ for some prime $q > p$ (2 points).
- Taking the equation $\mod q^2$ and getting $q^2|p^3 + 1$ (1 point).
- Concluding $q^2|p^2 - p + 1$ and finishing the problem (3 points).
- 0 points were awarded if only a correct answer was provided without any formal arguments.
- -1 point if a correct answer is missing.

2 B2

Problem: Aumann, Bill, and Charlie each roll a fair 6-sided die with sides labeled 1 through 6 and look at their individual rolls. Each flips a fair coin and, depending on the outcome, looks at the roll of either the player to his right or the person to his left, without anyone else knowing which die he observed. Then, at the same time, each of the three players states the expected value of the sum of the rolls based on the information he has. After hearing what everyone said, the three players again state the expected value of the sum of the rolls based on the information they have. Then, for the third time, after hearing what everyone said, the three players again state the expected value of the sum of the rolls based on the information they have. Prove that Aumann, Bill, and Charlie say the same number the third time.

Proposed by Eric Neyman.

Solution: Suppose that Aumann, Bill, and Charlie rolled the numbers a, b, and c, respectively. Call what Aumann, Bill, and Charlie say for the kth time A_k, B_k, and C_k (for $k = 1, 2, 3$). Note that A_1 is a plus whichever of b and c Aumann saw plus 3.5 (the expected value of the third roll), and so everyone can deduce the sum of the two numbers that Aumann saw, and similarly for Bob and Charlie. Case 1: It is not the case that $A_1 = B_1 = C_1$. Without loss of generality, assume that $A_1 \neq B_1$ and $A_1 \neq C_1$, and that Aumann saw b. Aumann knows that Bob saw c since otherwise we would have $A_1 = B_1$. Thus, Aumann can deduce $b + c$ and he knows a and b so he knows $a + b + c$. Everyone knows that $A_1 \neq B_1$ and $A_1 \neq C_1$, so everyone knows that Aumann knows $a + b + c$ after everyone says the first number. Thus,$A_3 = B_3 = C_3 = A_2$, as desired. Case 2: $A_1 = B_1 = C_1$. Subcase I: $a = b = c$. In this case, $A_3 = B_3 = C_3$ because everyone is in the same position (i.e. everyone sees their number and someone else's number and hears that everyone says the same thing both the first time and the second time). Subcase II: It is not the case that $a = b = c$. Note that in this case it can't be that everyone looked clockwise or everyone looked counterclockwise. Assume without loss of generality that Aumann and Bob looked at each other's rolls, and that Charlie looked at Aumann's roll. Then $a + b = a + c$, so $b = c$. Everyone saw two different rolls and thus knows that they are in this subcase. After everyone says their first number, Aumann sees two possibilities: either $c = b$, Charlie looked at Aumann's roll, and Bob looked at Auman's roll; or $c = a$ and Charlie looked at Bob's roll (in which case it doesn't matter where Bob looked). The second possibility is twice as likely, so Aumann says $\frac{1}{3}(a+b+b) + \frac{2}{3}(a+b+a) = \frac{5a+4b}{3}$.By analogy, Bob says $\frac{5b+4a}{3}$ and Charlie says $\frac{5c+4a}{3} = \frac{5b+4a}{3}$. Thus,$A2 \neq B2 = C2$, and so everyone finds out that Aumann's roll is the unique one. Thus, $A3 = B3 = C3 = a + 2b$,and we are done."

Rubric:

- Noting that each person's first statement is their number plus the number they saw plus 3.5 (1 point).

Approach 1: Casework on A_1, B_1, C_1. Award 1 further point for dealing with any case other than $A_1 = B_1 = C_1$, and 2 further points dealing with this case only (nonadditive with each other, and nonadditive if they deal with 2 subcases of the first case).

Approach 2: Casework on a, b, c. Award 1 point for dealing with the case a, b, c pairwise distinct, and 2 points for dealing with the case $a = b$ and c different (nonadditive with each other; no points for doing $a = b = c$).

Thus, the maximum score for an incomplete solution is 3 points.

3 B3

Problem: Let ABC be a triangle. Construct three circles k_1, k_2 and k_3 with the same radius such that they intersect each other at a common point O inside the triangle ABC and $k_1 \cap k_2 = \{A, O\}$,$k_2 \cap k_3 = \{B, O\}$,$k_3 \cap k_1 = \{C, O\}$. Let t_a be a common tangent of circles k_1 and k_2 such that A is closer to t_a than O. Define t_b and t_c similarly. Those three tangents determine a triangle MNP such that triangle ABC is inside the triangle MNP. Prove that the area of MNP is at least 9 times the area of ABC.

Solution: If the center are O_1, O_2, O_4 it's easy to see that $\triangle O_1O_2O_3 \sim \triangle ABC$ and that $\mathcal{H}(I, 1 + \frac{R}{r}) : \triangle O_1O_2O_3 \to \triangle MNP$, where R and r are circumcircle and incircle radii of $\triangle O_1O_2O_3$, hence $1 + \frac{R}{r} \geq 3$ so the area is at least 9 times larger.

Proposed by Marko Medvedev.

Rubric: Award 0 points for the following

- Claiming that $\triangle O_1O_2O_3 \equiv \triangle ABC$ without proof.
- Claiming that $\triangle MNP \sim \triangle ABC$ without proof.
- Claiming that $\triangle O_1O_2O_3 \sim \triangle MNP$ without proof.
- Finding angles and/or cyclic quadrilaterals without any intent.
- Claim parallel sides without any proof.
- Restating the problem in terms of some other triangles, etc.

Give 1 point, not additive with anything else if have the following

- Proving any one of the following $\triangle O_1O_2O_3 \sim \triangle MNP$ or ,$\triangle O_1O_2O_3 \sim \triangle ABC$, or $\triangle ABC \sim \triangle MNP$.

Give one point, additive for proving the following

- $\triangle O_1O_2O_3 \equiv \triangle ABC$ and $\triangle O_1O_2O_3 \sim \triangle MNP$
- Proving that MO_1,PO_3 and NO_2 intersect precisely at the incenter of $O_1O_2O_3$.
- Stating that there is a homothethy between MNP and $O_1O_2O_3$, and finding the coefficient.
- Proving that $R \geq 2r$. (2 points)
- Combining these statements. (2 points)

Deducting points

- Deduct 1 point for minor errors in proofs.
- Deduct 2 points for bigger errors.
- Deduct 1 if the final answer is wrong purely because of calculation errors.