1.1 [5] Find the number of pairs of real numbers (x, y) such that $x^4 + y^4 = 4xy - 2$.

1.2 [5] Define a function given the following 2 rules:

1) for prime p, $f(p) = p + 1$

2) for positive integers a and b, $f(ab) = f(a) \cdot f(b)$.

For how many positive integers $n \leq 100$ is $f(n)$ divisible by 3?

1.3 [5] Let a sequence be defined as follows: $a_0 = 1$, and for $n > 0$, a_n is $\frac{1}{3}a_{n-1}$ with probability $\frac{1}{2}$ and is $\frac{1}{9}a_{n-1}$ with probability $\frac{1}{2}$. What is the expected value of $\sum_{n=0}^{\infty} a_n$?

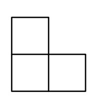
2.1 [7] Compute the period (i.e. length of the repeating part) of the decimal expansion of $\frac{1}{729}$.

2.2 [7] Let ABC be a triangle with side lengths $13, 14, 15$. The points on the interior of ABC with distance at least 1 from each side are shaded. The area of the shaded region can be written in simplest form as $\frac{m}{n}$. Find $m + n$.

2.3 [7] Sophie has 20 distinguishable pairs of socks in a laundry bag. She pulls them out one at a time. After pulling out 30 socks, the expected number of unmatched socks among the socks that she has pulled out can be expressed in simplest form as $\frac{m}{n}$. Find $m + n$.

Calculus 1. [5] Noted magician Casimir the Conjurer has an infinite chest full of weighted coins. For each $p \in [0, 1]$, there is exactly one coin with probability p of turning up heads. Kapil the Kingly draws a coin at random from Casimir the Conjurer's chest, and flips it 10 times. To the amazement of both, the coin lands heads up each time! On his next flip, if the expected probability that Kapil the Kingly flips a head is written in simplest form as $\frac{p}{q}$, then compute $p + q$.

Estimation 1. [7] A 2-by-2018 grid is completely covered by non-overlapping L-tiles (see diagram below) and 1-by-1 squares. If the L-tiles can be rotated and flipped, there are a total of M such tilings.



What is $\ln M$?

Give your answer as an integer or a decimal. If your answer is A and the correct answer is C, then your score will be $\max\{\lfloor 7.5 - \frac{|A-C|}{20}^{1.5} \rfloor, 0\}$.

Miscellaneous 1. [9] Consider all cubic polynomials $f(x)$ such that $f(2018) = 2018$, the graph of f intersects the y-axis at height 2018, the coefficients of f sum to 2018, and $f(2019) > f(2018)$.

We define the infimum of a set S as follows. Let L be the set of lower bounds of S. That is, $\ell \in L$ if and only if for all $s \in S$, $\ell \leq s$. Then the infimum of S is $\max(L)$.

Of all such $f(x)$, what is the infimum of the leading coefficient (the coefficient of the x^3 term)?

4.1 [9] The number 400000001 can be written as $p \cdot q$, where p and q are prime numbers. Find the sum of the prime factors of $p + q - 1$.

4.2 [9] Some number of regular polygons meet at a point on the plane, so that the polygons' interiors do not overlap, but the polygons fully surround the point (i.e. a sufficiently small circle centered at the point would be contained in the union of the polygons). What is the largest possible number of sides in any of the polygons?

4.3 [9] Let $0 \leq a, b, c, d \leq 10$. For how many ordered quadruples (a, b, c, d) is $ad - bc$ a multiple of 11?

5.1 [12] Let w and h be positive integers and define $N(w, h)$ to be the number of ways of arranging wh people of distinct heights for a photoshoot in such a way that they form w columns of h people, with the people in each column sorted by height (i.e. shortest at the front, tallest at the back). Find the largest value of $N(w, h)$ that divides 1008.

5.2 [12] Find x^2 given that $\tan^{-1} x + \tan^{-1} 3x = \frac{\pi}{6}$ and $0 < x < \frac{\pi}{6}$.

5.3 [12] Let k be the largest integer such that 2^k divides $\left(\prod_{n=1}^{25}\left(\sum_{i=0}^{n}\binom{n}{i}\right)^2\right)\left(\prod_{n=1}^{25}\left(\sum_{i=0}^{n}\binom{n}{i}^2\right)\right)$.

Find k.

Calculus 2. [9] Three friends are trying to meet for lunch at a cafe. Each friend will arrive independently, at random between 1:00 pm and 2:00 pm. Each friend will only wait for 5 minutes by themselves before leaving. However, if another friend arrives within those 5 minutes, the pair will wait 15 minutes from the time the second friend arrives. What is the probability the three friends meet for lunch?

Estimation 2. [12] How many perfect squares have the digits 1 through 9 each exactly once when written in base 10?

You must give your answer as a nonnegative integer. If your answer is A and the correct answer is C, your score will be $\lfloor 12.5 \cdot \min\{\left(\frac{A}{C}\right)^2, \left(\frac{C}{A}\right)^2\} \rfloor$.

Miscellaneous 2. [15] What is the sum of the possible values for the complex number a such that the coefficient of the x^5 term in the power series expansion of $\frac{x^3+ax^2+3x-4}{2x^2+ax+2}$ is 1?

7.1 [15] Find the number of nonzero terms of the polynomial $P(x)$ if

$$x^{2018} + x^{2017} + x^{2016} + x^{999} + 1 = (x^4 + x^3 + x^2 + x + 1)P(x).$$

7.2 [15] Compute the smallest positive integer n that is a multiple of 29 with the property that for every positive integer k that is relatively prime to n, $k^n \equiv 1 \pmod{n}$.

7.3 [15] Kite $ABCD$ has right angles at B and D, and $AB < BC$. Points $E \in AB$ and $F \in AD$ satisfy $AE = 4, EF = 7, FA = 5$. If $AB = 8$ and point X lies outside $ABCD$ while satisfying $XE - XF = 1$ and $XE + XF + 2XA = 23$, then XA may be written as $\frac{a-b\sqrt{c}}{d}$ for a, b, c, d positive integers with $\gcd(a^2, b^2, c, d^2) = 1$ and c squarefree. Find $a + b + c + d$.

8.1 [18] Let a, b, and c be such that the coefficient of the $x^a y^b z^c$-term in the expansion of $(x+2y+3z)^{100}$ is maximal (no other term has a strictly larger coefficient). Find the sum of all possible values of $1,000,000a + 1,000b + c$.

8.2 [18] The triangle ABC satisfies $\overline{AB} = 10$, and has angles $\angle A = 75°$, $\angle B = 60°$, $\angle C = 45°$. Let I_A be the center of the excircle opposite A, and let D, E be the circumcenters of triangles BCI_A and ACI_A respectively. If O is the circumcenter of triangle ABC, then the area of triangle EOD can be written as $\frac{a\sqrt{b}}{c}$ for square-free b and coprime a, c. Find the value of $a+b+c$.

8.3 [18] If a and b are positive integers such that $3\sqrt{2+\sqrt{2+\sqrt{3}}} = a\cos\frac{\pi}{b}$, find $a+b$.

Calculus 3. [15] Let $\mathcal{R}(f(x))$ denote the number of distinct real roots of $f(x)$. Compute $\sum_{a=1}^{1009}\sum_{b=1010}^{2018}\mathcal{R}(x^{2018} - ax^{2016} + b)$.

Estimation 3. [18] Andrew starts with the 2018-tuple of binary digits $(0, 0, \ldots, 0)$. On each turn, he randomly chooses one index (between 1 and 2018) and flips the digit at that index (makes it 1 if it was a 0 and vice versa). What is the smallest k such that, after k steps, the expected number of ones in the sequence is greater than 1008?

You must give your answer as a nonnegative integer. If your answer is A and the correct answer is C, then your score will be $\max\{\lfloor 18.5 - \frac{|A-C|}{40}^{1.8}\rfloor, 0\}$.

Miscellaneous 3. [20] Suppose $x, y \in \mathbb{Z}$ satisfy:

$$y^4 + 8x^3 + 6y^2 + 32x + 1 = (x^2 - y^2)(x^2 + y^2 + 24)$$

Find the sum of all possible values of $|xy|$.