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Number Theory A Solutions

1. Find the sum of all prime numbers p such that p divides

$$(p^2+p+20)^{p^2+p+2}+4(p^2+p+22)^{p^2-p+4}.$$

Proposed by Sunay Joshi

Answer: $\boxed{344}$

We claim that the primes are $p = 2, 61, 281$, yielding an answer of $2 + 61 + 281 = 344$. First, the expression is congruent to $20^4 + 4 \cdot 22^4$ modulo p by Fermat's Little Theorem. Next, note that by the Sophie-Germain Identity, we can rewrite the expression as $2^4 \cdot (10^4 + 4 \cdot 11^4) = 2^4 \cdot (10^2 + 2 \cdot 11^2 - 2 \cdot 10 \cdot 11)(10^2 + 2 \cdot 11^2 + 2 \cdot 10 \cdot 11)$, which equals $2^6 \cdot 61 \cdot 281$. Since p divides this product, p must be among $\{2, 61, 281\}$, and the result follows.

2. Compute the sum of all positive integers whose positive divisors sum to 186.

Proposed by Nancy Xu

Answer: $\boxed{202}$

The sum of the divisors of an integer with prime factorization $p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$ is given by $(1 + p_1 + \dots p_1^{n_1})(1 + p_2 + \dots p_2^{n_2}) \dots (1 + p_k + \dots p_k^{n_k})$. We see that $186 = 2 \cdot 3 \cdot 31$, so it has factors $1, 2, 3, 6, 31, 62, 93, 186$. It is clear that 1 and 2 cannot be written as the sum of powers of a prime, so by trying out small primes, the only remaining possibilities are $186 = 6 \cdot 31 = (1 + 5)(1 + 2 + 4 + 8 + 16)$ and $186 = 3 \cdot 62 = (1 + 2)(1 + 61)$. Thus our two numbers are $5 \cdot 16 = 80$ and $2 \cdot 61 = 122$, sum the sum is $80 + 122 = 202$.

3. Given $k \geq 1$, let p_k denote the k-th smallest prime number. If N is the number of ordered 4-tuples (a, b, c, d) of positive integers satisfying $abcd = \prod_{k=1}^{2023} p_k$ with $a < b$ and $c < d$, find $N \pmod{1000}$.

Proposed by Sunay Joshi

Answer: $\boxed{112}$

We claim that if $n \geq 2$ is square-free, then the number of ordered 4-tuples (a, b, c, d) satisfying $abcd = n$ with $a < b$ and $c < d$ is exactly $\frac{1}{4}\tau(n)^2 - \frac{1}{2}\tau(n)$. To see this, note that a 4-tuple (a, b, c, d) coresponds to a choice of divisor $d_1 = ab$ of n. By symmetry, there are $\frac{\tau(d_1)}{2}$ ways to pick the pair (a, b) with $a < b$. Similarly there are $\frac{\tau(n/d_1)}{2}$ ways to pick (c, d) with $c < d$. Therefore the total number of 4-tuples is $(\sum_{d_1 | n} \frac{\tau(d_1)}{2} \frac{\tau(n/d_1)}{2}) - 2 \cdot \frac{\tau(1)}{2} \cdot \frac{\tau(n)}{2}$, where we subtract the terms corresponding to $d_1 = 1, n$. Since n is square-free, we have $\gcd(d_1, n/d_1) = 1$, hence $\tau(d_1)\tau(n/d_1) = \tau(n)$ and the above reduces to $\frac{1}{4}\tau(n)^2 - \frac{1}{2}\tau(n)$, as claimed.

Returning to the problem, note that for $n = \prod_{k=1}^{2023} p_k$, we have $\tau(n) = 2^{2023}$, hence $N = 2^{2 \cdot 2023 - 2} - 2^{2023-1} = 2^{2022}(2^{2022} - 1)$. This is clearly $0 \pmod 8$. By Euler's Theorem, we see that $N \equiv 2^{22}(2^{22} - 1) \equiv 48^2(48^2 - 1) \equiv 112 \pmod{125}$. By the Chinese Remainder Theorem, $N \equiv 112 \pmod{1000}$, our answer.

4. Find the number of ordered pairs (x, y) of integers with $0 \leq x < 2023$ and $0 \leq y < 2023$ such that $y^3 \equiv x^2 \pmod{2023}$.

Proposed by Brandon Cho

Answer: $\boxed{3927}$

Since $2023 = 7 \cdot 17^2$, by the Chinese Remainder Theorem it suffices to consider the pair of congruences $y^3 \equiv x^2 \pmod 7$ and $y^3 \equiv x^2 \pmod{17^2}$.

For the former, note that since $x^2 \in \{0, 1, 2, 4\}$ and $y^3 \in \{0, 1, -1\}$, we must have $y^3 \equiv x^2 \equiv 0$ or $y^3 \equiv x^2 \equiv 1$. The former corresponds to $(0, 0)$. The latter is satisfied when $x \in \{1, -1\}$ and $y \in \{1, 2, 4\}$. This yields 6 pairs. Thus this case has 7 solutions.

For the latter congruence, we consider two cases. The first case is when 17 does not divide y, so that 17 does not divide x. Further the map $y \mapsto y^3$ is a bijection of the set of units of $\mathbb{Z}/17^2\mathbb{Z}$. Therefore each choice of unit x corresponds to a unique solution for y. Since there are $17^2 - 17$ units mod 17^2, we have a total of $17^2 - 17$ pairs in this case. The second case is when 17 divides y, hence 17 divides x. Any such pair (x, y) satisfies the congruence since both sides are 0. It follows that there are $17 \cdot 17$ pairs in this third case. Summing, we find $2 \cdot 17^2 - 17$ pairs.

Finally, we multiply the number of solutions to each of the two congruences to find an answer of $7 \cdot (2 \cdot 17^2 - 17) = 3927$.

5. A positive integer $\ell \geq 2$ is called *sweet* if there exists a positive integer $n \geq 10$ such that when the leftmost nonzero decimal digit of n is deleted, the resulting number m satisfies $n = m\ell$. Let S denote the set of all sweet numbers ℓ. If the sum $\sum_{\ell \in S} \frac{1}{\ell - 1}$ can be written as $\frac{A}{B}$ for relatively prime positive integers A, B, find $A + B$.

   Proposed by Sunay Joshi

   Answer: $\boxed{71}$

   Let $\nu_p(t)$ denote the highest power of the prime p dividing t. We claim that $\ell \geq 2$ is sweet iff: (i) all prime factors of $\ell - 1$ are elements of $\{2, 3, 5, 7\}$, (ii) $\nu_3(\ell - 1) \leq 2$, (iii) $\nu_7(\ell - 1) \leq 1$, (iv) $3 \cdot 7$ does not divide $\ell - 1$, and (v) $\ell - 1 \neq 1, 3, 7, 9$. To see this, suppose that $n = m\ell$, where m is the number obtained by deleting the leftmost digit of n. Write $n = 10^k a + b$, where $a \in \{1, \ldots, 9\}$ is the leftmost digit of n, so that $m = b$. Then $n = m\ell$ is equivalent to $10^k a + b = \ell b$, or $(\ell - 1)b = 10^k a$ for some k-digit number b.

   The condition $(\ell - 1)b = 10^k a$ for an arbitrary positive integer b is equivalent to $(\ell - 1) | 10^k a$ for some $a \in \{1, \ldots, 9\}$, which is equivalent to the first four conditions above.

   If $\ell - 1 \geq 10$, then $(\ell - 1)b = 10^k a$ for some k-digit number b is equivalent to $(\ell - 1) | 10^k a$, since the equality forces b to have at most k digits: $b \leq 10^k \cdot 9/10 < 10^k$. If $\ell - 1 \in \{1, \ldots, 9\}$, then in the cases $\ell - 1 \in \{1, 3, 7, 9\}$, b must have at least k digits. The value of b in each case is at least $\frac{10^k}{1}$, $\frac{3 \cdot 10^k}{3}$, $\frac{7 \cdot 10^k}{7}$, and $\frac{9 \cdot 10^k}{9}$, respectively.

   Thus $\ell \in S$ iff the five conditions above hold. In terms of prime factorization, $\ell \in S$ iff $\ell - 1 \neq 1, 3, 7, 9$ and $\ell - 1 = 2^x 5^y 3^z 7^w$, where $x \geq 0$, $y \geq 0$, and $(z, w) \in \{(0, 0), (0, 1), (1, 0), (2, 0)\}$. Splitting the desired sum into a product over primes, we find

$$\sum_{\ell \in S} \frac{1}{\ell - 1} = \left(\sum_{x \geq 0} \frac{1}{2^x} \right) \left(\sum_{y \geq 0} \frac{1}{5^y} \right) \left(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{7} \right) - \left(\frac{1}{1} + \frac{1}{3} + \frac{1}{7} + \frac{1}{9}\right),$$

   where we subtract terms corresponding to the cases $\ell - 1 = 1, 3, 7, 9$. By the geometric series formula, this equals $\frac{250}{63} - \frac{100}{63} = \frac{50}{21}$. Thus our answer is $50 + 21 = 71$.

6. Given a positive integer ℓ, define the sequence $\{a_n^{(\ell)}\}_{n=1}^{\infty}$ such that $a_n^{(\ell)} = \lfloor n + \sqrt[\ell]{n} + \frac{1}{2} \rfloor$ for all positive integers n. Let S denote the set of positive integers that appear in all three of the sequences $\{a_n^{(2)}\}_{n=1}^{\infty}$, $\{a_n^{(3)}\}_{n=1}^{\infty}$, and $\{a_n^{(4)}\}_{n=1}^{\infty}$. Find the sum of the elements of S that lie in the interval $[1, 100]$.

   Proposed by Sunay Joshi

Answer: $\boxed{4451}$

We claim that a number $k+1$ is skipped by the sequence $\{a_n^{(\ell)}\}_{n=1}^{\infty}$ iff $k+1 = m + \lceil (m+\frac{1}{2})^\ell \rceil$ for some $m \geq 0$. To see this, suppose $k+1$ is skipped by the sequence, so that $a_n = k$ and $a_{n+1} \geq k+2$. The condition $a_n = k$ is equivalent to $k \leq n + \sqrt[\ell]{n} + \frac{1}{2} < k+1$ and thus $(m-\frac{1}{2})^k \leq n < (m+\frac{1}{2})^\ell$, where $m = k - n$. The condition $a_{n+1} \geq k+2$ is equivalent to $k+2 \leq (n+1) + \sqrt[\ell]{n+1} + \frac{1}{2}$, which can be rewritten as $(m+\frac{1}{2})^\ell - 1 \leq n$. Combining these two inequality chains, we find that $n = \lceil (m+\frac{1}{2})^\ell \rceil - 1$, hence the skipped number is $k+1 = m + \lceil (m+\frac{1}{2})^\ell \rceil$, as claimed.

It follows that the numbers skipped in the sequence for $\ell = 2$ are $m + \lceil m^2 + m + \frac{1}{4} \rceil = (m+1)^2$; the numbers skipped for $\ell = 3$ are $m + \lceil m^3 + \frac{3}{2}m^2 + \frac{3}{4}m + \frac{1}{8} \rceil = m + m^3 + \lceil \frac{3}{2}m^2 + \frac{3}{4}m + \frac{1}{8} \rceil$; and the numbers skipped for $\ell = 4$ are $m + \lceil m^4 + 2m^3 + \frac{3}{2}m^2 + \frac{1}{2}m + \frac{1}{16} \rceil = m + m^4 + 2m^3 + \lceil \frac{3}{2}m^2 + \frac{1}{2}m + \frac{1}{16} \rceil$. The skipped numbers for $\ell = 2$ are $1, 4, 9, 16, 25, 36, 49, 64, 81, 100$, the skipped numbers for $\ell = 3$ are $1, 5, 18, 46, 96$, and the skipped numbers for $\ell = 4$ are $1, 7, 42$. The sum of (distinct) numbers that are skipped in at least one of the sequences can be seen to be 599, hence the sum of the numbers in $[1, 100]$ that are not skipped in any list is $5050 - 599 = 4451$, our answer.

7. For a positive integer n, let $f(n)$ be the number of integers m satisfying $0 \leq m \leq n-1$ such that there exists an integer solution to the congruence $x^2 \equiv m \pmod{n}$. It is given that as k goes to ∞, the value of $f(225^k)/225^k$ converges to some rational number p/q, where p, q are relatively prime positive integers. Find $p+q$.

Proposed by Frank Lu

Answer: $\boxed{37}$

First, suppose that m, n are relatively prime. Then, notice that for every pair of residues $a \pmod{m}$ and $b \pmod{n}$, if $x^2 \equiv a \pmod{m}$ and $x^2 \equiv b \pmod{n}$ both have solutions, then the corresponding residue r modulo mn (through using Chinese Remainder Theorem) is such that $x^2 \equiv r \pmod{mn}$. Similarly, if there is a solution for this residue, then there is such a solution for the residues of $r \pmod{m}$ and $r \pmod{n}$. Therefore, $f(mn) = f(m)f(n)$. It thus suffices for us to compute $f(5^{2k})$ and $f(3^{2k})$. We will perform this computation in generality for a prime p.

Suppose that $x^2 \equiv b \pmod{p^{2k}}$ has a solution, where $0 \leq b < p^{2k}$. Then, notice that if b is divisible by p, then it is divisible by p^2. From here, it follows that, writing $b = b'p^2$, we must have a solution to $x^2 \equiv b' \pmod{p^{2k}}$. Therefore, using this logic, $f(p^{2k})$ is equal to the sum of the number of residues b relatively prime to p so that $x^2 \equiv b \pmod{p^{2i}}$ has a solution, for $0 \leq i \leq k$.

For $i = 0$ there is exactly one such solution, namely $b \equiv 1 \pmod{1}$. Now, we claim that there are $p^{2i-1}(p-1)/2$ such solutions for i. To show this, we inductively argue the following: given $p > 2$ is a prime and b relatively prime to p, if $x^2 \equiv b \pmod{p^i}$ has a solution, then $x^2 \equiv b + cp^i \pmod{p^{i+1}}$ has a solution for $c = 0, 1, \ldots, p-1$. Indeed, observe that, given x' so $x'^2 \equiv b \pmod{p^i}$, suppose that $x'^2 \equiv b + ap^i \pmod{p^{i+1}}$. Then, $(x' + dp^i)^2 \equiv b + ap^i + 2x'dp^i \pmod{p^{i+1}}$. For this to equal $b + cp^i$ modulo p^{i+1}, we need for $x'd + a \equiv c \pmod{p}$. But as b is relatively prime to p, so is x'; therefore this has exactly one such solution.

In particular, this means that p^{i+1} has p times as many residues b satisfying the above condition than p^i. So recalling that for p there are $(p-1)/2$ such residues, it follows that p^i has $p^{i-1}(p-1)/2$ such residues.

From here, we compute that $f(p^{2k}) = 1 + \frac{p-1}{2} \sum_{i=0}^{k-1} p^{2i+1}$. Therefore, note that $f(p^{2k})/p^{2k}$ equals

$p^{-2k} + \frac{p-1}{2}\sum_{i=0}^{k-1} p^{2i+1-2k} = p^{-2k} + \frac{p-1}{2}\sum_{i=0}^{k-1} p^{-2i-1}$. As k goes to ∞, this becomes $\frac{p-1}{2}\sum_{i=0}^{\infty} p^{-2i-1} = \frac{p-1}{2}\frac{p^{-1}}{1-p^{-2}} = \frac{p}{2(p+1)}$.

Our desired fraction, by multiplying, is thus $\frac{3}{8}\frac{5}{12} = \frac{5}{32}$, so $p+q = 37$.

8. For $n \geq 2$, let $\omega(n)$ denote the number of distinct prime factors of n. We set $\omega(1) = 0$. Compute the absolute value of

$$\sum_{n=1}^{160} (-1)^{\omega(n)} \left\lfloor \frac{160}{n} \right\rfloor.$$

Proposed by Julian Shah

Answer: $\boxed{22}$

$\lfloor \frac{160}{n} \rfloor$ counts the number of multiples of n less than or equal to 160. Instead of summing over multiples of integers less than 160, we can sum over divisors of integers less than 160:

$$\sum_{n=1}^{160} (-1)^{\Omega(n)} \left\lfloor \frac{160}{n} \right\rfloor = \sum_{n=1}^{160} \sum_{d|n} (-1)^{\Omega(d)}$$

Note that since $f(n) = (-1)^{\Omega(n)}$ is multiplicative, the function $g(n) = \sum_{d|n} f(n)$ is also multiplicative. We can see that $g(p^k) = -(k-1)$ for any prime p; in particular, $g(p) = 0$. Thus g vanishes on any n that has a prime divisor with exponent 1, and we can ignore all such integers in computing the sum $\sum_{n=1}^{160} g(n)$. The integers from 1 to 160 that have no prime divisor of exponent 1 are generated multiplicatively by: the prime powers $\{4, 8, 16, 32, 64, 128\}$, $\{9, 27, 81\}$, $\{25, 125\}$, $\{49\}$, and $\{121\}$. We see that most of these generators can't be multiplied by anything else without exceeding 160. Thus we are then left to do casework on the generators $\{4, 8, 16\}$, $\{9, 27\}$, and $\{25\}$, which is much simpler. This yields the additional values of $4 \cdot 9 = 36$, $4 \cdot 27 = 108$, $4 \cdot 25 = 100$, $8 \cdot 9 = 72$, $16 \cdot 9 = 144$. We must also include $n = 1$ in our sum. Summing, we find an value of $1 + (-1-2-3-4-5-6) + (-1-2-3) + (-1-2) + (-1) + (-1) + (1+2+1+2+3) = -22$, so our answer is 22.