PUMaC

Number Theory B

1. Find the number of positive integers $n < 100$ such that $\gcd(n^2, 2023) \neq \gcd(n, 2023^2)$.

2. I have a four-digit palindrome $\underline{a}\,\underline{b}\,\underline{b}\,\underline{a}$ that is divisible by b and is also divisible by the two-digit number $\underline{b}\,\underline{b}$. Find the number of palindromes satisfying both of these properties.

3. Find the integer x for which $135^3 + 138^3 = x^3 - 1$.

4. A number is called *good* if it can be written as the sum of the squares of three consecutive positive integers. A number is called *excellent* if it can be written as the sum of the squares of four consecutive positive integers. (For instance, $14 = 1^2 + 2^2 + 3^2$ is good and $30 = 1^2 + 2^2 + 3^2 + 4^2$ is excellent.) A good number G is called *splendid* if there exists an excellent number E such that $3G - E = 2025$. If the sum of all splendid numbers is S, find the remainder when S is divided by 1000.

5. Call an arrangement of n not necessarily distinct nonnegative integers in a circle *wholesome* when, for any subset of the integers such that no pair of them is adjacent in the circle, their average is an integer. Over all wholesome arrangements of n integers where at least two of them are distinct, let $M(n)$ denote the smallest possible value for the maximum of the integers in the arrangement. What is the largest integer $n < 2023$ such that $M(n+1)$ is strictly greater than $M(n)$?

6. What is the smallest possible sum of six distinct positive integers for which the sum of any five of them is prime?

7. You play a game where you and an adversarial opponent take turns writing down positive integers on a chalkboard; the only condition is that, if m and n are written consecutively on the board, $\gcd(m, n)$ must be squarefree. If your objective is to make sure as many integers as possible that are strictly less than 404 end up on the board (and your opponent is trying to minimize this quantity), how many more such integers can you guarantee will eventually be written on the board if you get to move first as opposed to when your opponent gets to move first?

8. How many positive integers $n \leq \operatorname{lcm}(1, 2, \ldots, 100)$ have the property that n gives different remainders when divided by each of $2, 3, \ldots, 100$?

Name:

Team:

Write answers in table below:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8