PUMaC 2015

Geometry A

1. [**3**] For her daughter's 12th birthday, Ingrid decides to bake a dodecagon pie in celebration. Unfortunately, the store does not sell dodecagon shaped pie pans, so Ingrid bakes a circular pie first and then trims off the sides in a way such that she gets the largest regular dodecagon possible. If the original pie was 8 inches in diameter, the area of pie that she has to trim off can be represented in square inches as $a\pi - b$ where a, b are integers. What is $a + b$?

2. [**3**] Terry the Tiger lives on a cube-shaped world with edge length 2. Thus he walks on the outer surface. He is tied, with a leash of length 2, to a post located at the center of one of the faces of the cube. The surface area of the region that Terry can roam on the cube can be represented as $\frac{p\pi}{q} + a\sqrt{b} + c$ for integers a, b, c, p, q where no integer square greater than 1 divides b, p and q are coprime, and $q > 0$. What is $p + q + a + b + c$? (Terry can be at a location if the shortest distance along the surface of the cube between that point and the post is less than or equal to 2.)

3. [**4**] Cyclic quadrilateral $ABCD$ satisfies $\angle ADC = 2 \cdot \angle BAD = 80^\circ$ and $\overline{BC} = \overline{CD}$. Let the angle bisector of $\angle BCD$ meet AD at P. What is the measure, in degrees, of $\angle BPD$?

4. [**4**] Find the largest r such that 4 balls each of radius r can be packed into a regular tetrahedron with side length 1. In a packing, each ball lies outside every other ball, and every ball lies inside the boundaries of the tetrahedron. If r can be expressed in the form $\frac{\sqrt{a}+b}{c}$ where a, b, c are integers such that $\gcd(b, c) = 1$, what is $a + b + c$?

5. [**5**] Let P, A, B, C be points on circle O such that C does not lie on arc $\widehat{BAP}$, $\overline{PA} = 21$, $\overline{PB} = 56$, $\overline{PC} = 35$ and $m\angle BPC = 60^\circ$. Now choose point D on the circle such that C does not lie on arc $\widehat{BDP}$ and $\overline{BD} = 39$. What is $\overline{AD}$?

6. [**6**] Triangle ABC is inscribed in a unit circle ω. Let H be its orthocenter and D be the foot of the perpendicular from A to BC. Let $\triangle XYZ$ be the triangle formed by drawing the tangents to ω at A, B, C. If $AH = HD$ and the side lengths of $\triangle XYZ$ form an arithmetic sequence, the area of $\triangle ABC$ can be expressed in the form $\frac{p}{q}$ for relatively prime positive integers p, q. What is $p + q$?

7. [**7**] Triangle ABC has $\overline{AB} = \overline{AC} = 20$ and $\overline{BC} = 15$. Let D be the point in $\triangle ABC$ such that $\triangle ADB \sim \triangle BDC$. Let l be a line through A and let BD and CD intersect l at P and Q, respectively. Let the circumcircles of $\triangle BDQ$ and $\triangle CDP$ intersect at X. The area of the locus of X as l varies can be expressed in the form $\frac{p}{q}\pi$ for positive coprime integers p and q. What is $p + q$?

8. [**8**] The incircle of acute triangle ABC touches BC, AC, and AB at points D, E, and F, respectively. Let P be the second intersection of line AD and the incircle. The line through P tangent to the incircle intersects AB and AC at points M and N, respectively. Given that $\overline{AB} = 8$, $\overline{AC} = 10$, and $\overline{AN} = 4$, let $\overline{AM} = \frac{a}{b}$ where a and b are positive coprime integers. What is $a + b$?