PUMaC

Wave 1. Set 1.1. Regular.

1.1.1 [5] If $x - y = x^2 - y^2 = 12$, what is $x + y$?

1.1.2 [5] Let $ABCD$ be a rectangle of side lengths 3 and 4. Imagine folding the rectangle over the diagonal BD. The area of the new shape can be written with integers a, b in the form $\frac{a}{b}$, where $\gcd(a, b) = 1$. What is $a + b$? (If there is overlapping area, it should only be considered once.)

1.1.3 [5] A flea hops around the edge of a circular table that has a circumference of 2016 cm. The length of the flea's hop is n cm, where n is an integer between 1 and 2016, inclusive, and the flea repeatedly hops around the circumference in length n cm circular arcs. Find the sum of all n for which the flea needs exactly 2016 hops to return to its starting point for the first time.

1.1.4 [5] Let $a_1 = a_2 = \cdots = a_{100} = 1$ and for $n > 100$, $a_n = a_{n-100} + a_{n-99}$. For $n \geq 1$, let $d_n = a_{n+1} - a_n$. If $d_k = 1$ and $d_{k+2} = 2016$, compute k.

Wave 1. Set 1.2. Regular.

1.2.1 [7] Heesu draws an equilateral triangle ABC with side 1 m on a large piece of paper, then throws a dart at it. The dart lies within 1 m of two of A, B, or C (it could be inside or outside the triangle). The probability that it lies within 1m of all three vertices given it lies within 1 m of two can be expressed in the form $\frac{a}{a+\pi}$, where $a = b\pi - c\sqrt{d}$. What is $b + c + d$?

1.2.2 [7] Compute the largest positive integer n for which there are exactly three two-digit positive integers k such that $\frac{n}{k}$ is a two-digit positive integer.

1.2.3 [7] A *flashlight* with angle measure θ is a device that can be positioned at any point P on the coordinate plane and it illuminates everything in some angle of measure θ from vertex P. Flashlights of angle measure 60° are positioned at $(0, 0)$ and $(0, 2)$ so that they shine directly at each other. Let R be the region of the plane where a flashlight of angle measure 135° could be positioned so that every flashlight is illuminated by every other flashlight. The area of R can be written in reduced form as $\frac{a\sqrt{b}-c\pi}{d}$. Compute $a + b + c + d$.

1.2.4 [7] A random sequence is generated as follows: $a_0 = 1$, and for each k, a_k is a random multiple of a_{k-1} such that a_k is a divisor of 2016 (for convention, note that n is both a multiple and a divisor of n). (For example, a_1 is randomly chosen among all divisors of 2016 that are multiples of 1.) If the expected number of factors of a_3 is $\frac{m}{n}$, compute $m + n$.

Wave 1. Set 1.3. Exotic.

Estimation 1. Find the best rational approximation x to $\sqrt{0.7}$ such that $|x - \sqrt{0.7}|$ is as small as possible. You may either find an $x = \frac{a}{b}$ where a, b are coprime integers or find a decimal approximation.

Let C be the actual answer and A be the answer you submit. Your score will be given by $\left\lceil 2 + \frac{3.3}{0.1+e^{30|A-C|}} \right\rceil$, where $\lceil x \rceil$ denotes the smallest integer which is $\geq x$.

Calculus 1. [5] For what positive value k does the equation $\ln x = kx^2$ have exactly one solution?

Miscellaneous 1. How many words were in the Power Round?

Let C be the actual answer and A be the answer you submit. Your score will be 5 if $A \in [C/5, 5C]$ and 0 if A is not in the range.

Wave 2. Set 2.1. Regular.

2.1.1 [10] What is the sum of all the real parts of the solutions to the complex equation $z^2 + \overline{z} = 0$?

2.1.2 [10] An *arithmo-geometric sequence* is a sequence of integers $a_0, a_1, a_2, \ldots$ such that $a_0 = 0$ and for every positive integer k, $a_{2k-2}, a_{2k-1}, a_{2k}$ form an arithmetic sequence and $a_{2k-1}, a_{2k}, a_{2k+1}$ form a geometric sequence. Compute the number of arithmo-geometric sequences containing 2016.

2.1.3 [10] How many tetrahedra have side lengths of 2,3,4,5,6,7?

2.1.4 [10] Triangle ABC has circumradius 201 and inradius 6. Three circles of radius $\frac{p}{q}$, where p and q are relatively prime positive integers, meet at exactly one point, and each is tangent to two distinct sides of triangle ABC. Find $p + q$.

Wave 2. Set 2.2. Regular.

2.2.1 [12] Let a and b be integers such that $1 \leq a \leq 20$ and $1 \leq b \leq 16$. Suppose a valid Pin (Princeton identification number) is any 3 digit number that does not start with 0. If a and b are randomly chosen, the probability that ab is a Pin in reduced form is $\frac{m}{n}$. What is $m + n$?

2.2.2 [12] How many integer solutions are there between 0 and 90 inclusive to the equation

$$x^3 + 64x^2 - x + 27 \equiv 0 \pmod{91}?$$

2.2.3 [12] Compute the sum of the four smallest positive integers k such that there is a pair of points with nonnegative integer x-coordinates on the parabola $y = \frac{1}{k}x^2 + 2x$ such that the distance between the two points is rational.

2.2.4 [12] We are given a circle and a square in the plane with radius r and side length s, respectively. Let X be the locus of points that is a midpoint between some point inside the circle and some point inside the square. If the minimum value of $[X]/rs$, where $[X]$ is the area of region X, can be written as $\frac{a+\sqrt{\pi}}{b}$, find $a + b$.

Wave 2. Set 2.3. Exotic.

Estimation 2. A lattice point is a coordinate pair (a, b) where both a, b are integers. What is the number of lattice points (x, y) that satisfy $\frac{x^2}{2016} + \frac{2y^2}{2016} < 1$ and $y \equiv 2x \pmod 7$?

Let C be the actual answer, A be the answer you submit. Your score will be given by

$$\max\left(0, \lceil 10 - e^{\frac{|A-C|}{100}} \rceil\right).$$

Calculus 2. [10] Compute

$$\int_0^{2016} \frac{x - 1008}{x^2 - 2016x - 2017}\, dx.$$

Miscellaneous 2. Suppose you play a game of poker with four players where everyone is equally skilled. The likelihood therefore of any specific person of beating *any set of other players* is their proportion of chip stack. The four players have stack sizes of 10,000, 10,000, 20,000, and 25,000 chips and the payout for finishing 4th, 3rd, 2nd, and 1st are respectively \$200, \$300, \$500, and \$1000. What is the expected payout for the person currently in first place with a stack size of 25,000? (For example, the probability that a person with stack size 10,000 winning first is $\frac{10000}{65000}$.)

Let C be the actual answer and A be the answer you submit. Your score will be given by

$$\max\left(0, \lceil 10 \cdot (1 - \frac{|A-C|}{C}) \rceil\right).$$

Wave 3. Set 3.1. Regular.

3.1.1 [15] What is $2^{2016} \pmod{211}$ in reduced form?

3.1.2 [15] Imagine a paper cutout in the shape of an equilateral triangle that has three other equilateral triangles that each share a distinct side of the original triangle. This shape can be folded up into a tetrahedron. How many such shapes in 2D are there that fold into a tetrahedron? Valid shapes must consist of a tessellation of equal sized equilateral triangles, and two shapes are considered the same (and shouldn't be counted separately) if they can be rotated or flipped to match.

3.1.3 [15] Suppose that $\triangle ABC$ has side lengths $AB = 2013$, $AC = 2015$ and $BC = 1007$. Let P be a point inside $\triangle ABC$. Let X and Y be the feet of the perpendiculars from P to AB and AC respectively. Suppose that $\angle BPX = \angle CPY$. Let the perpendicular bisector of XY intersect segment BC at Q. If $\left(\frac{BQ}{QC}\right)^2 = \frac{a}{b}$ where a, b are coprime positive integers, find $a + b$.

3.1.4 [15] Rem and Ram play a game on a large 2016-by-2016 chessboard. First, they randomly choose a cell to put their Subaru in, then randomly choose who goes first. Rem is allowed to move their Subaru up, down, left, or right by 1 square, while Ram is allowed to move their Subaru up 1 right 2 or up 2 right 1 (like a knight moving towards the upper-right). They play until Ram cannot make a move: if the piece is in the upper-right corner, then Rem wins; otherwise, Ram wins. The probability that Rem wins can be expressed in the form $\frac{p}{q}$, where $(p, q) = 1$. Find $p + q$.

Wave 3. Set 3.2. Regular.

3.2.1 [20] David adds 448 2x2x2 cubes to a 8x8x8 cube to form a 16x16x16 cube. If the 8x8x8 cube is colored differently on each face but the 2x2x2 cubes are indistinguishable upon rotation and reflection and from each other, how many ways can David do this?

3.2.2 [20] Let f be the cubic polynomial defined as

$$f(x) = 64\frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)} + 27\frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)} + 8\frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)} + \frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)}.$$

What is the remainder when $f(1738)$ is divided by 1000?

3.2.3 [20] Let the sequences $a_1, a_2, a_3, \dots$ and $b_1, b_2, b_3, \dots$ be defined by $a_1 = \frac{1}{7}$, $b_1 = \frac{1}{3}$ and $a_{i+1} = a_i b_i$, $b_{i+1} = \frac{a_i b_i}{2a_i b_i - a_i - b_i + 1}$. If $a_{100} = \frac{p}{q}$ where p, q are coprime positive integers, find the remainder when $p + q$ is divided by 100.

3.2.4 [20] Given an n-dimensional hyper-rectangle with side lengths $s_1, s_2, \dots, s_n$, define the content of the hypercube to be the product $s_1 s_2 \dots s_n$ (so, in particular, a 0-dimensional hypercube, or point, has content 1), and define the gumption of the hypercube to be the sum of the contents of every m-dimensional hypercube that makes up its boundary, for $0 \le m \le n$. For instance, a cube with side lengths 1, 2, and 3 has content $1 \times 2 \times 3 = 6$ and gumption $6 + 2(1 \times 2 + 1 \times 3 + 2 \times 3) + 4(1 + 2 + 3) + 8 = 60$, since it contains 1 3-dimensional cube, 6 2-dimensional squares, 12 1-dimensional sides, and 8 0-dimensional points.

Alice draws an n-dimensional hypercube with integer side lengths and $n \geq 2$. She makes sure to write down its gumption, 1008, but she forgets to label her drawing with the lengths of each side. Hoping to figure out what they were, she passes the picture to Bob, a perfect mathematician, who merely glances at the gumption and the dimension before confidently declaring that he knows what all of the side lengths were. What was n?

Wave 3. Set 3.3. Exotic.

Estimation 3. Let a sequence be defined by $a_0 = a_1 = a_2 = a_3 = 1$, and for $n \geq 4$,

$$a_n = \frac{a_{n-1} \cdot a_{n-3} + a_{n-2}^2}{a_{n-4}}.$$

Suppose in scientific notation, $x \cdot 10^m = a_{20} + a_{16}$, where $1 \leq x < 10$ and m is an integer. What is m?

Let C be the actual answer, A be the answer you submit, and $D = |A - C|$. Your score will be given by

$$\max\left(0, 15 - D\right).$$

Calculus 3. [15] Consider the curves in 3-space with equations $x^n + y^n + z^n = 1$ for $n \in \mathbb{N}$. Denote A_{2k} the area of the surface using $n = 2k$. What is

$$\lim_{k\to\infty} A_{2k}?$$

Miscellaneous 3. What is the 2016th smallest prime number?

Let C be the actual answer and A be the answer you submit. Your score will be given by

$$\max\left(0, \lceil 15 \cdot (1 - \frac{|A-C|}{C}) \rceil\right).$$