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# Team Round Solutions

1. Consider a 2021-by-2021 board of unit squares. For some integer $k$, we say the board is *tiled* by $k$-by-$k$ squares if it is completely covered by (possibly overlapping) $k$-by-$k$ squares with their corners on the corners of the unit squares. What is the largest integer $k$ such that the minimum number of $k$-by-$k$ squares needed to tile the 2021-by-2021 board is exactly equal to 100?

   *Proposed by: Ollie Thakar*

   **Answer:** $\boxed{224}$

   Consider the set $S$ of unit squares in the $(a, b)$ position on the 2021-by-2021 board where $a$ and $b$ are both congruent to 1 modulo $k$. If $2021 = mk + r$, with $0 < r \leq k$, then there are $(m+1)^2$ elements of $S$. Each $k$-by-$k$ square in the tiling covers precisely one of these elements of $S$, and it is easy to see that by establishing a regular pattern, we can tile the whole board with $(m+1)^2$ of the $k$-by-$k$ squares.

   Thus, we must find which $k$ gives $m = 9$, the largest of which is $k = 224$.

2. Gary is baking cakes, one at a time. However, Gary's not been having much success, and each failed cake will cause him to slowly lose his patience, until eventually he gives up. Initially, a failed cake has a probability of 0 of making him give up. Each cake has a $\frac{1}{2}$ of turning out well, with each cake independent of every other cake. If two consecutive cakes turn out well, the probability resets to 0 immediately after the second cake. On the other hand, if the cake fails, assuming that he doesn't give up at this cake, his probability of breaking on the next failed cake goes from $p$ to $p + 0.5$. If the expected number of successful cakes Gary will bake until he gives up is $\frac{p}{q}$, for relatively prime $p, q$, find $p + q$.

   *Proposed by: Frank Lu*

   **Answer:** $\boxed{86}$

   Let $f(p, c)$ be the function giving the expected number of cakes Gary will bake until he gives up, given that his probability of giving up after the next failed cake is currently $p$, and his last $c$ cakes were successful.

   Now, note that $f(p, 0) = \frac{1}{2} + \frac{1}{2}(1-p)f(p+0.5, 0) + \frac{1}{2}f(p, 1)$, and $f(p, c) = \frac{1}{2} + \frac{1}{2}(1-p)f(p+0.5, 0) + \frac{1}{2}f(0, c+1)$ for $c \geq 1$. Note that this equation is not defined for $c > 1$ if $p \neq 0$, and both equations are only defined for $0 \leq p \leq 0.9$. For $p = 1$, we see that the first equation is now $f(1, 0) = \frac{1}{2}(1 + f(1, 1))$, and the second equation is $f(1, 1) = \frac{1}{2}(1 + f(0, 2))$.

   We also observe that $f(0, c)$ is constant for $c \geq 2$, since Gary's probability of giving up at a given time depends only on the last two cakes he attempted to bake, as well as his probability of giving up right before baking his last cake (so Gary baking more than 2 successes in the row doesn't alter the expected number of cakes he bakes afterwards). Thus, we see that $f(0, c) = 1 + f(0.5, 0)$ for $c \geq 2$, from the second equation.

   Now, note that combining these two equations yields us that $f(p, 0) = \frac{3}{4} + \frac{3}{4}(1-p)f(p+0.5, 0) + \frac{1}{4}f(0, 2)$. But then we see that, solving the system yields us that $f(0, 0) = \frac{3}{4} + \frac{3}{4}f(0.5, 0) + \frac{1}{4}(1 + f(0.5, 0))$, or that $f(0, 0) = 1 + f(0.5, 0)$, and similarly we see that $f(0.5, 0) = \frac{3}{4} + \frac{3}{4}\frac{1}{2}f(1, 0) + \frac{1}{4}(1 + f(0.5, 0))$, or that $f(0.5, 0) = \frac{4}{3} + \frac{1}{2}f(1, 0)$, and that $f(1, 0) = 1 + \frac{1}{4}f(0.5, 0)$. The last two equations yield us in turn that $f(0.5, 0) = \frac{11}{6} + \frac{1}{8}f(0.5, 0)$, or that $f(0.5, 0) = \frac{44}{21}$, which in turn means that $f(0, 0) = \frac{65}{21}$, yielding an answer of 86.

3. Alice and Bob are playing a guessing game. Bob is thinking of a number $n$ of the form $2^a 3^b$, where $a$ and $b$ are positive integers between 1 and 2020, inclusive. Each turn, Alice guess a number $m$, and Bob will tell her either $\gcd(m, n)$ or $\text{lcm}(m, n)$ (letting her know that he is

saying that gcd or lcm), as well as whether any of the respective powers match up in their prime factorization. In particular, if $m = n$, Bob will let Alice know this, and the game is over. Determine the smallest number $k$ so that Alice is always able to find $n$ within $k$ guesses, regardless of Bob's number or choice of revealing either the lcm, or the gcd .

*Proposed by: Frank Lu*

**Answer:** $\boxed{11}$

We can consider how $a$ lies in the range $\{1, 2, \ldots, 2020\}$, as does $b$. Let $k(x, y)$ be the number of guesses it takes, where $a$ lies in $\{1, 2, \ldots, x\}$, and $b$ lies in $\{1, 2, \ldots, y\}$. We first make the observation that $k(x, y) = k(y, x)$, by symmetry: Alice can just use the same strategy, but flipping the exponents on $x, y$. From here, assume WLOG that $x \geq y$.

First, we claim that $k(x, 1) = \lfloor \log_2 x \rfloor + 1$, which we will later refer to as the 1D case (a diagram can illustrate why). To see this, note that every time Alice guesses a number, either Bob will reveal what it is, or Alice will be told that the exponent on 2 is larger or smaller. Effectively, then, the result will be like starting over from a completely different game with a narrower range. Hence, we see that $k(x, 1)$ is the minimum of $\max(k(y-1, 1), k(x-y-1, 1))$, for $y$ between 1 and $x$, and letting $k(0, 1) = 0$. An inductive argument can be used here to show this (like a binary search tree, effectively).

We now claim that $k(x, y) = \lfloor \log_2 \max(x, y) \rfloor + 1$, by inducting on the maximum of $x, y$, with strong induction. Our base case for 1 is given.

Now, given that we've shown this where $\max(x, y) \leq i - 1$, consider $k(x, y)$, where their maximum is $i \geq 2$ and WLOG say $x = i$. We know that $k(x, y) \geq \lfloor \log_2 x \rfloor + 1 = k(x, 1)$, since any strategy that Alice can use to guarantee in $k(x, y)$ steps can be applied for the game with only the first exponent varying. We will now show the other direction. To do this, have Alice first guess $2^{\lfloor (i+1)/2 \rfloor} 3^{\lfloor (y+1)/2 \rfloor}$. First, if Alice guessed one of the exponents correctly, then note that the set of possible values reduces down to the 1-D case (as though Alice and Bob were playing with only one exponent varying), which Alice can guarantee in at most $k(i, 1)$ guesses. Otherwise, if Bob reports that the gcd or lcm of his number and Alice's is $2^{\lfloor (i+1)/2 \rfloor} 3^{\lfloor (y+1)/2 \rfloor}$, then note that the set of possible values is at most $i/2$ for the exponents for 2 and $y/2$ for those of 3. Strong induction yields that, from here, at most $\lfloor \log_2 i/2 \rfloor + 1$ guesses are needed, which means that in total at most $\lfloor \log_2 i \rfloor + 1$ guesses are needed.

Finally, if Bob reports that the exponents are different, but gives a gcd or lcm that isn't the number itself, then we see that whichever exponents differ from Alice's guess are the exponents for Bob's number. From here, Alice can play as though she was in the 1D case, with a range of exponents that is again at most $i/2$. In all of these cases, we see that $k(x, y) \leq \lfloor \log_2 x \rfloor + 1 = k(x, 1)$, which in turn yields us the equality, as desired.

Finally, we see that our answer is just $\lfloor \log_2 2020 \rfloor + 1 = 11$.

4. Find the number of points $P \in \mathbb{Z}^2$ that satisfy the following two conditions:

   1) If $Q$ is a point on the circle of radius $\sqrt{2020}$ centered at the origin such that the line $\overline{PQ}$ is tangent to the circle at $Q$, then $\overline{PQ}$ has integral length. 2) The $x$-coordinate of $P$ is 38.

   *Proposed by: Ollie Thakar*

   **Answer:** $\boxed{16}$

   Notice that $38^2 + 24^2 = 2020$. Then, let $P$ have coordinates $(38, y)$, and label the length of $\overline{PQ}$ as $T$. For now, we will only deal with positive $y$. We know from power of a point theorem that $(y + 24)(y - 24) = T^2$. Re-arranging this expression gives us $(y + T)(y - T) = 24^2 = 2^6 \cdot 3^2$.

   Now, we know that $y+T$ and $y-T$ must be integer factors of $2^6 \cdot 3^2$. There are $(6+1)(2+1) = 21$ factors of $2^6 \cdot 3^2$, of which 20 come in pairs and 1 is a perfect square. Thus, there are 11 pairs of factors multiplying to $2^6 \cdot 3^2$.

Of those pairs of factors, 3 pairs have 1 odd factor and 1 even factor, while the remaining pairs have 2 even factors. $(y+T)(y-T) = 2^6 \cdot 3^2$ means that the only the pairs with 2 even factors will lead to integer values of $T$ and $y$. Each factor pair leads to a unique solution pair of $y$ and $T$. Thus, there are 8 possibilities for $y$, when $y > 0$. Then, there are 8 more possibilities for $y$ that are negative, so the total is 16.

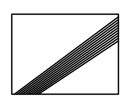
*Note: We also accepted the answer of* $\boxed{14}$ *since it isn't clear that* $P$ *is allowed to be taken on the circle and still yield a valid configuration.*

5. Suppose two polygons may be glued together at an edge if and only if corresponding edges of the same length are made to coincide. A $3 \times 4$ rectangle is cut into $n$ pieces by making straight line cuts. What is the minimum value of $n$ so that it's possible to cut the pieces in such a way that they may be glued together two at a time into a polygon with perimeter at least 2021?

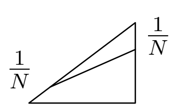
   *Proposed by: Austen Mazenko*

   **Answer:** $\boxed{202}$

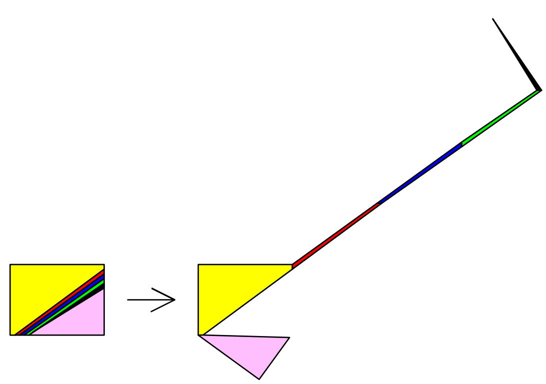
   For $n$ pieces, edges must be glued together at least $n-1$ times, and each gluing event reduces the overall perimeter by twice the length of the edges being glued together. Furthermore, every time a cut is made to divide the bar into more pieces, it increases the total perimeter by at most twice the length of the largest cut, which is 5 (the length of the rectangle's diagonal). To form $n$ pieces, there are at most $n-1$ cuts. Hence, an upper bound for the perimeter is $3+4+3+4+2\cdot 5\cdot(n-1) - 2\cdot 0\cdot(n-1) = 10n+4$ since every edge being glued together has a length $> 0$ and all cuts have length $\leq 5$. Accordingly, we need $10n+4 \geq 2021 \implies n \geq 202$ since $n$ must be an integer. To see that $n = 202$ is sufficient, put the bar on the coordinate plane so that it has one vertex on the origin and one at $(4,3)$. First, make 200 cuts from $(\frac{i}{N}, 0)$ to $(4, 3-\frac{i}{N})$ for $1 \leq i \leq 200$ and some large integer $N$.



Finally, cut the bottom right triangle like so:




Now, all of the thin strips have two edges of length $\frac{1}{N}$, so they may be glued together in sequence like so:



By Pythagorean Theorem, each cut has length at least $\sqrt{\left(3-\frac{201}{N}\right)^2+\left(4-\frac{201}{N}\right)^2}-\frac{1}{N}$. Making $N$ arbitrarily large, each cut may have a length sufficiently close to 5 and each small edge may have sufficiently small length so that the perimeter will exceed 2021, as desired.

6. We say that a string of digits from 0 to 9 is *valid* if the following conditions hold: First, for $2 \leq k \leq 4$, no consecutive run of $k$ digits sums to a multiple of 10. Second, between any two 0s, there are at least 3 other digits. Find the last four digits of the number of valid strings of length 2020.

   *Proposed by: Frank Lu*

   **Answer:** $\boxed{9040}$

   Let $a_l$ be the number of valid strings of length $l$ whose last digit is 0, and define $b_l$ to be those whose second to last digit is 0, $c_l$ third to last digit is 0, and $d_l$ to be all other valid strings. Let $t_l = a_l + b_l + c_l + d_l$.

   Then, observe that we can construct the following recurrences:

   First, $a_l = d_{l-1}$, as for any valid string where there is no 0 in the last three digits, we can append a 0 to get a valid string. This holds for $l \geq 2$.

   Next, $b_l = 7a_{l-1}$. To see this, suppose we have a valid string ending with a 0 of length $l-1$, whose last three digits are $x, y, 0$. Then, we can add any digit except for 0, and the digits equivalent to $-y$ and $-x-y \pmod{10}$, all of which are distinct. By a similar logic, $c_l = 7b_{l-1}$. Note, however, that these equations only hold for $l \geq 4$.

   Finally, we note that $d_l = 7c_{l-1} + 6d_{l-1}$, by applying a similar logic. Summing all of these up, we see that $t_l = 7t_{l-1}$. We do, however, need to compute $t_3$ first, as we've seen that this recurrence only holds for $l \geq 4$. We compute: $a_1 = 1, d_1 = 9, b_1 = c_1 = 0$. For $l = 2 : a_2 = 9, b_2 = 9, c_2 = 0, d_2 = 72$, and for $l = 3 : a_3 = 72, b_3 = 72, c_3 = 72, d_3 = 504$. This yields that $t_1 = 10, t_2 = 90, t_3 = 720$, which gives us that for $l \geq 4, t_l = 720 \cdot 7^{l-3}$, which gives us that $t_{2020} = 720 \cdot 7^{2017}$.

   Now, to compute the last four digits: we see that this is equivalent to $0 \pmod{16}$, so we need to find what it is $\pmod{625}$. Note that $7^{500} \equiv 1 \pmod{625}$, by Euler totient, which gives us that $t_{2020} \equiv 95 \cdot 7^{17} \pmod{625}$. But as this is divisible by 5, we can just find what $19 \cdot 7^{17} \pmod{125}$ is. However, we see that $7^4 = 2401 \equiv 25 + 1 \pmod{125}$, so hence $7^{16} \equiv (25+1)^4 \equiv 1 + 25 \cdot 4 \equiv 101 \pmod{125}$. But then we have that $19 \cdot 7^{17} \equiv 133 \cdot 101 \equiv 58 \pmod{125}$, implying that $t_{2020} \equiv 290 \pmod{625}$. Noting that we have that $t_{2020}$ is divisible by 16 yields us that $t_{2020}$'s last four digits are 9040.

7. Let $X$, $Y$, and $Z$ be concentric circles with radii 1, 13, and 22, respectively. Draw points $A$, $B$, and $C$ on $X$, $Y$, and $Z$, respectively, such that the area of triangle $ABC$ is as large as possible. If the area of the triangle is $\Delta$, find $\Delta^2$.

   *Proposed by: Daniel Carter*

   **Answer:** $\boxed{24300}$

   Let the circles be centered at the origin $O$ and without loss of generality $A = (1, 0)$. Consider fixing $A$ and $B$ and letting $C$ vary. The area of the triangle is maximized when the height from $C$ onto $AB$ is perpendicular to the tangent of $Z$ at $C$, or in other words when $CO$ is perpendicular to $AB$. Likewise we have $AO$ is perpendicular to $BC$, so $B$ and $C$ have the same $x$-coordinate. Let $B = (x, b)$ and $C = (x, c)$ with $x$ and $b$ negative and $c$ positive.

   Then the circle equations give $x^2 + b^2 = 169$ and $x^2 + c^2 = 484$, and $CO \perp AB$ gives $x(x-1) + bc = 0$. Solve the first two equations for $b$ and $c$ and plug into the third to give $x(x-1) + \sqrt{(169 - x^2)(484 - x^2)} = 0$. Rearranging, squaring, and simplifying gives the cubic $x^3 - 327x^2 + 40898 = 0$. We know $x$ is negative, so we can look for a root of the form $-n$

where $n$ is a factor of $40898 = 2 \cdot 11^2 \cdot 13^2$. We don't need to try many to find the solution $x = -11$. Then $b = -4\sqrt{3}$, $c = 11\sqrt{3}$, and the area of the triangle is $90\sqrt{3} = \sqrt{24300}$.

8. Let there be a tiger, William, at the origin. William leaps 1 unit in a random direction, then leaps 2 units in a random direction, and so forth until he leaps 15 units in a random direction to celebrate PUMaC's 15th year.

   There exists a circle centered at the origin such that the probability that William is contained in the circle (assume William is a point) is exactly $\frac{1}{2}$ after the 15 leaps. The area of that circle can be written as $A\pi$. What is $A$?

   *Proposed by: Aditya Gollapudi*

   **Answer:** $\boxed{1240}$

   Let $D = \{\theta_1, \theta_2, ..., \theta_{15}\}$ represent the random directions that William has selected. Then the point that William is at can be represented by $(\sum_{i=1}^{15} i\cos(\theta_i), \sum_{i=1}^{15} i\sin(\theta_i))$. Thus the area of the smallest circle containing is $\pi((\sum_{i=1}^{15} i\cos(\theta_i))^2 + (\sum_{i=1}^{15} i\sin(\theta_i))^2)$ and we need only solve for $(\sum_{i=1}^{15} i\cos(\theta_i))^2 + (\sum_{i=1}^{15} i\sin(\theta_i))^2$ Expanding this out we get $\sum_{i=1}^{15} i^2\cos(\theta_i)^2 + \sum_{i=1}^{15} i^2\sin(\theta_i)^2 + \sum_{i=1}^{15}\sum_{j=1,j\neq i}^{15} ij\cos(\theta_i+\theta_j) + \sum_{i=1}^{15}\sum_{j=1,j\neq i}^{15} ij\sin(\theta_i+\theta_j)$ which can be simplified to $\sum_{i=1}^{15} i^2 + \sum_{j=1,j\neq i}^{15}\cos(\theta_i+\theta_j)$ By the symmetry of the distribution and the symmetry of cos, the second term is less than zero $\frac{1}{2}$ of the time and greater than zero $\frac{1}{2}$ half of the time. Thus the area of the circle in which William is contained $\frac{1}{2}$ of the time is simply $\pi\sum_{i=1}^{15} i^2$ which is well known to be $\pi\frac{15*16*31}{6} = 1240\pi$

9. Consider a regular 2020-gon circumscribed into a circle of radius 1. Given three vertices of this polygon such that they form an isosceles triangle, let $X$ be the expected area of the isosceles triangle they create. $X$ can be written as $\frac{1}{m\tan((2\pi)/n)}$ where $m$ and $n$ are integers. Compute $m+n$.

   *Proposed by: Ollie Thakar*

   **Answer:** $\boxed{5049}$

   Draw radii from the center of the circumcircle to each vertex of the isosceles triangle. If the central angles thus created are $\alpha, \alpha, 2\pi - 2\alpha$ then the area is simply $\sin\alpha - \frac{1}{2}\sin(2\alpha)$. This can be seen with law of sines. Let the original side lengths of the triangle be $A, B, C$, and angles be $a, b, c$. Then, because the center divides the triangle into three sub-triangles, the subtriangles have areas $\frac{1}{2}RC\cos c$, $\frac{1}{2}RB\cos b$, and $\frac{1}{2}RA\cos a$, which I found by dividing them into two congruent right triangles and using base times height. Law of sines tells us, however, that $A = 2R\sin a, B = 2R\sin b, C = 2R\sin c$. Plugging these relations into our area formula, remembering also that $R = 1$ and $\cos a \sin a = \frac{1}{2}\sin(2a)$ tells us that the total area of the triangle is $Area = \frac{1}{2}(\sin(2a) + \sin(2b) + \sin(2c))$, which is the formula that I used to get $\sin\alpha - \frac{1}{2}\sin(2\alpha)$.

   For each vertex, the combined area of all of the isosceles triangles whose distinct angle lies at that vertex is simply the sum of $\sin\alpha - \frac{1}{2}\sin(2\alpha)$ where $\alpha \in \{\frac{2\pi}{2020}, \frac{4\pi}{2020}, ..., \frac{2018\pi}{2020}\}$.

   The sum of $\sin\alpha$ for $\alpha$ in the above range is just the height of the regular 2020-gon with side-length 1, which is $h = \frac{1}{\tan(\pi/2020)}$.

   The sum of $\sin(2\alpha)$ for $\alpha$ in the above range is the imaginary part of the sum $1 + z + ... + z^{1009}$ where $z$ is the 1010th root of unity, so is clearly 0.

   The total number of isosceles triangles is $1009 * 2020$, and the sum of all of their areas, by our above logic, is $2020 * \frac{1}{2\tan(\pi/2020)}$, so the expected area of one of the triangles is $\frac{1}{1009\tan(\pi/2020)}$.

10. Let $N$ be the number of sequences of positive integers greater than 1 where the product of all of the terms of the sequence is $12^{64}$. If $N$ can be expressed as $a(2^b)$, where $a$ is an odd positive integer, determine $b$.

*Proposed by: Frank Lu*

**Answer:** $\boxed{128}$

Let $g(n)$ be the number of ordered tuples of any size so that the entries multiply to $n$, and all are positive integers that are at least 2. Let $f(n)$ be the sum over all such ordered tuples of the sum of the entries in the tuples. For sake of convenience, we set $g(1) = 1$, representing how we have the empty product of a tuple of length 0, and similarly we have that $f(1) = 0$, representing the empty sum.

Then, we see that, summing over possible first entries, yields us that $g(n) = \sum_{d|n, d\neq n} g(d)$. We'll write this as $2g(n) = \sum_{d|n} g(d)$ We also know that $g(p) = 1$ for any prime $p$, as we have that the only ordered tuple is $(p)$.

By a similar logic, we can see that $g(p^k) = 2^{k-1}$, by using an inductive argument.

Now, observe that, given $n = p^a q^b$, where $p, q$ are distinct primes and $a, b \geq 1$, that we have that $2g(n) = g(n) + \sum_{d|n} g(d) = g(n) + \sum_{d|n/p} g(d) + \sum_{d|n/q} g(d) - \sum_{d|n/pq} g(d) = 2g(n/p) + 2g(n/q) - 2g(n/pq)$. We thus see that $g(n) = 2g(n/p) + 2g(n/q) - 2g(n/pq)$, unless $n = pq$, where in this case we have that $g(pq) = 3$.

Now, let $f_b(a) = \frac{g(p^a q^b)}{2^{a-1}}$. Note then that our recurrence relation becomes $2^a f_{b+1}(a+1) = 2^a f_{b+1}(a) + 2^{a+1} f_b(a+1) - 2^a f_b(a)$, or that $f_{b+1}(a+1) - f_{b+1}(a) = 2f_b(a+1) - f_b(a)$, unless we have both $a, b$ equaling zero, where then we have that $f_1(1) = 3$.

This yields us, for $a, b$ where at least one of $a, b$ is greater than 1, that $f_b(a) - f_b(0) = \sum_{i=0}^{a-1} 2f_{b-1}(i+1) - f_{b-1}(i)$. But $f_b(0) = g(q^b)$, which for $b > 1$ is equal to $2^{b-1}$, so in fact this becomes $f_b(a) - f_b(0) = f_{b-1}(a) - f_{b-1}(0) + \sum_{i=1}^{a} f_{b-1}(i)$. But we can then rewrite this as $f_b(a) = f_{b-1}(a) + \sum_{i=0}^{a} 2f_{b-1}(i)$, for all positive integers $a$ and for $b > 1$, and for $b = 1, a > 1$. We can thus see that, with $f_0(a) = 1$ for $a \neq 0$ that $f_1(a) = a + 2$. Note that $f_b$ will be a degree $b$ polynomial.

Now, suppose we can write $f_b(a) = \sum_{i=0}^{b} c_i \binom{a+i}{i}$, for some coefficients $i$. It thus follows that $f_{b+1}(a) = \sum_{i=0}^{b} c_i \binom{a+i}{i} + \sum_{j=0}^{a} \sum_{i=0}^{b} c_i \binom{j+i}{i} = \sum_{i=0}^{b} c_i \binom{a+i}{i} + \sum_{i=0}^{b} c_i \binom{a+i+1}{i+1} = c_0 \binom{a}{0} + c_b \binom{a+b+1}{b+1} + \sum_{i=1}^{b} (c_i + c_{i-1}) \binom{a+i}{i}$.

But starting with the fact that the coefficients begin as $1, 1$, with $a+2 = a+1+1$, it thus follows that we have that $f_b(a) = \sum_{i=0}^{b} \binom{b}{i} \binom{a+i}{i}$, giving us in turn that $g(p^a q^b) = 2^{a-1} (\sum_{i=0}^{b} \binom{b}{i} \binom{a+i}{i})$.

Applying this to our situation, we want to evaluate $f(2^{128} 3^{64})$. We can express this then as $2^{127} (\sum_{i=0}^{64} \binom{64}{i} \binom{128+i}{i})$. Note that in this sum inner sum, we have two terms whose largest power of 2 dividing them is 1, namely $i = 0, i = 64$, and one term whose largest power of 2 dividing them is 2, namely $\binom{64}{32} \binom{128+32}{32}$. Their sum is $\binom{196}{64} + 1$. But note that this is divisible by 4, yielding us the answer 128. To see this, consider the product $(192 * 191 * 190 * \ldots * 129)/(64 * 63 * \ldots * 1)$. Observe then that we can pair elements together in the denominator by $i$ and $64 - i$, and pairing $i$ and $320 - i$, save for the elements $192, 64, 160, 32$. Only one of these is equivalent to $3 \pmod 4$ when the largest power of 2 is divided out. This is then equivalent to $3 \pmod 4$, showing that $\binom{196}{64} + 1$ is at least divisible by 4.

11. Three (not necessarily distinct) points in the plane which have integer coordinates between 1 and 2020, inclusive, are chosen uniformly at random. The probability that the area of the

triangle with these three vertices is an integer is $\frac{a}{b}$ in lowest terms. If the three points are collinear, the area of the degenerate triangle is 0. Find $a+b$.

*Proposed by: Daniel Carter*

**Answer:** $\boxed{13}$

Let the three points be $(x_i, y_i)$ for $i \in \{1,2,3\}$. By the shoelace area formula, the area of the triangle is $|x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3|/2$, so it is an integer if the numerator is even. Considering the numerator mod 2, shifting any of the $x_i$ or $y_i$ by 2 at a time preserves the parity of the numerator. Add or subtract an even number from each of $x_2$, $y_2$, $x_3$, and $y_3$ to make $x_2'$, $y_2'$, etc. so that $x_2'$ and $x_3'$ are either $x_1$ or $x_1+1$ and $y_2'$ and $y_3'$ are either $y_1$ or $y_1+1$. If the resulting triangle has integer area, so did the original.

Note that there are an equal number of even numbers as odd numbers between 1 and 2020 inclusive. Thus the probability that $x_2' = x_1$ is $1/2$, and likewise for the other coordinates and possibilities. Out of the sixteen possibilities for $x_2'$, $y_2'$, $x_3'$, and $y_3'$, six of them form a right triangle with area $1/2$, and ten of them form a degenerate triangle with area 0. Thus the probability the original triangle had integer area is $10/16 = 5/8$, so the answer is $5+8=13$.

12. Given a sequence $a_0, a_1, a_2, \ldots, a_n$, let its *arithmetic approximant* be the arithmetic sequence $b_0, b_1, \ldots, b_n$ that minimizes the quantity $\sum_{i=0}^{n}(b_i - a_i)^2$, and denote this quantity the sequence's *anti-arithmeticity*. Denote the number of integer sequences whose arithmetic approximant is the sequence $4, 8, 12, 16$ and whose anti-arithmeticity is at most 20.

*Proposed by: Frank Lu*

**Answer:** $\boxed{15}$

First, we find a formula for the anti-arithmeticity for a sequence $a_0, a_1, a_2, a_3$, as well to find what the arithmetic sequence should be. Suppose we have arithmetic sequence $a-3d, a-d, a+d, a+3d$. Then, we see that the value of $\sum_{i=0}^{3}(a+(2i-3)d-a_i)^2$ can be evaluated to equal $4a^2 + 20d^2 - 2(a_0+a_1+a_2+a_3)a-$ $(6a_3+2a_2-2a_1-6a_0)d + a_0^2 + a_1^2 + a_2^2 + a_3^2$. We can re-write this as equal to $4(a - \frac{a_0+a_1+a_2+a_3}{4})^2 + a_0^2 + a_1^2 + a_2^2 + a_3^2 - 4(\frac{a_0+a_1+a_2+a_3}{4})^2 - 20(d - \frac{3a_3+a_2-a_1-3a_0}{20})^2 - 20(\frac{3a_3+a_2-a_1-3a_0}{20})^2$. We see then that the minimal value of this is equal to $a_0^2 + a_1^2 + a_2^2 + a_3^2 - 4(\frac{a_0+a_1+a_2+a_3}{4})^2 - 20(\frac{3a_3+a_2-a_1-3a_0}{20})^2$.

There are two ways to continue. Either through algebraic manipulation or via linear algebra arguments with orthogonal vectors $(1,-1,-1,1)$ and $(1,-3,3,-1)$, we see that this is equal to $(a_3 - a_2 - a_1 + a_0)^2/4 + (a_3 - 3a_2 + 3a_1 - a_0)^2/20$.

Now, note that we are given that $a_0 + a_1 + a_2 + a_3 = 40, 3a_3 + a_2 - a_1 - 3a_0 = 40$. But we see that that the anti-arithmeticity needs to be an integer. But let $a_3 - 3a_2 + 3a_1 - a_0 = s$, and let $a_3 - a_2 - a_1 + a_0 = t$. We can then see that $a_0 + a_3 = 20 + t/2, a_3 - a_0 = 12 + s/10$, and similarly we see that $a_1 + a_2 = 20 - t/2, a_2 - a_1 = 4 - 3s/10$, which requires us to have $s$ divisible by 10, and $t$ divisible by 2, and $s/10, t/2$ to have the same parity. We make the substitution $s/10 = a, t/2 = b$ to get that our anti-arithmeticity value is just $a^2 + 5b^2$, with $a, b$ having the same parity.

For the values to be at most 20, we can just enumerate: $(0,0), (0,\pm 2), (\pm 1, \pm 1), (\pm 2, 0), (\pm 3, \pm 1), (\pm 4, 0)$. The total number of pairs: $1+2+4+2+4+2 = 15$.

13. Will and Lucas are playing a game. Will claims that he has a polynomial $f$ with integer coefficients in mind, but Lucas doesn't believe him. To see if Will is lying, Lucas asks him on minute $i$ for the value of $f(i)$, starting from minute 1. If Will is telling the truth, he will

report $f(i)$. Otherwise, he will randomly and uniformly pick a positive integer from the range $[1,(i+1)!]$. Now, Lucas is able to tell whether or not the values that Will has given are possible immediately, and will call out Will if this occurs. If Will is lying, say the probability that Will makes it to round 20 is $\frac{a}{b}$. If the prime factorization of $b$ is $p_1^{e_1}\dots p_k^{e_k}$, determine the sum $\sum_{i=1}^{k} e_i$.

*Proposed by: Frank Lu*

**Answer:** $\boxed{289}$

Suppose Will has given the values $a_1, a_2, \dots, a_n$. Given that Will has lasted up to turn $n$, there is a polynomial so that $p$ so that $p(i) = a_i$ for each $i$. Furthermore, if $q$ is also a polynomial where this is possible, then we have that $p(i) - q(i)$ is divisible by $(i-1)(i-2)\dots(i-n)$. But by integer coefficients, we have that $p(i) = z(i-1)(i-2)\dots(i-n) + q(i)$. Thus, it follows that Will has one unique possible value of $a_{n+1}$ modulo $n!$ that works, which means he has a $\frac{1}{n!}$ chance of making it to the next round. Furthermore, the probability that he makes it past minute 2 is 1 (any line will work). Thus, the probability that he makes it to round $n$ is equal to $p = \prod_{i=1}^{n-1}\frac{1}{i!}$, given that he is lying. Now, we need to determine the prime valuations for each of the primes between 1 and 20. For a given prime $p$, this is equal to $\sum_{i=1}^{19}\sum_{k=1}^{\infty}\lfloor\frac{i}{p^k}\rfloor = \sum_{k=1}^{\infty}\sum_{i=1}^{19}\lfloor\frac{i}{p^k}\rfloor$. For $p = 11, 13, 17, 19$, this expression is just equal to $20-p$. For $p = 7$, this equals $7*1+6*2 = 19$, and for $p = 5$ this is $5*1+5*2+5*3 = 30$. For $p = 3$, the sum evaluates to $(3*(1+\dots+5)+2*6)+(9*1+2*2) = 45+25 = 70$. Finally, for $p = 2$, this is $2*(1+\dots+9)+4*(1+\dots+3+4)+8*1+2*4+4*1 = 90+40+8+8+4 = 150$. The total sum is thus $1+3+7+9+19+30+70+150 = 20+49+220 = 289$.

14. Let $N$ be the number of convex 27-gons up to rotation there are such that each side has length 1 and each angle is a multiple of $2\pi/81$. Find the remainder when $N$ is divided by 23.

*Proposed by: Michael Gintz and Rahul Saha*

**Answer:** $\boxed{12}$

Let us consider the roots of unity. Every such polygon can be constructed by taking some subset of the roots of unity which adds to 0, and each such subset uniquely defines a polygon. Two define the same polynomial if they are rotations of each other. We wish to show that the subsets are only those made by unioning equilateral triangles. Consider some subset that works. Letting $\omega$ be the smallest primitive root we have some polynomial in $\omega$ which has a root at $\omega$. Then since the cyclotomic polynomial is the minimal polynomial of $\omega$, this new polynomial must be a multiple of that. However, the cyclotomic polynomial of powers of primes is known to be $1 + x^{27} + x^{54}$, so our set of roots must contain equilateral triangles. Thus we can consider whether we have the first 27 roots. Two polygons will be equivalent iff these binary strings of length 27 are equivalent by rotation. Since this is a 27 gon there are 9 ones so by https://math.stackexchange.com/questions/721783/number-of-unique-sequences-with-circular-shifts our answer is

$$\frac{1}{27}\sum_{d|9}\phi(d)\binom{a/d+b/d}{a/d}$$

$$\frac{1}{27}\left(1\binom{27}{9}+2\binom{9}{3}+6\binom{3}{1}\right)\equiv 12 \pmod{23}.$$

*Note: Since the desired polygons in this problem are impossible, due to the condition on the angles, we also accepted the answer of* $\boxed{0}$.

15 Suppose that $f$ is a function $f:\mathbb{R}_{\geq 0}\to\mathbb{R}$ so that for all $x, y \in \mathbb{R}_{\geq 0}$ (nonnegative reals) we have that $f(x)+f(y) = f(x+y+xy)+f(x)f(y)$. Given that $f(\frac{3}{5}) = \frac{1}{2}$ and $f(1) = 3$, determine $\lfloor\log_2(-f(10^{2021}-1))\rfloor$.

*Proposed by: Frank Lu*

**Answer:** $\boxed{10104}$

First, we simplify down our functional equation for $f$. Notice that, using Simon's Favorite Factoring trick, we may write this as $(1-f(x))(1-f(y)) = 1-f(x+y+xy)$. We can then simplify down our function by writing $g(x) = 1 - f(x)$, yielding the function $g(x)g(y) = g(x+y+xy)$. Now, notice that, letting $h(x) = g(x-1)$, this is equivalent to writing $h(x+1)h(y+1) = h(x+y+xy+1)$. But then, notice that this is equivalent to writing $h(x)h(y) = h(xy)$ for all $x, y$ that are real and at least 1, the domain of $h$. From here, notice that the values of $h$ that we have are $h(\frac{8}{5}) = g(\frac{3}{5}) = 1 - f(\frac{3}{5}) = \frac{1}{2}$, and similarly that $h(2) = g(1) = 1 - f(1) = -2$. Now, notice then that $h(5) = h(8)/h(\frac{8}{5}) = -8/\frac{1}{2} = -16$. Therefore, we see that $h(10^{2021})$, by the multiplicativity, equals $h(10)^{2021} = (-32)^{2021} = 2^{10105}$. Therefore, it follows that $f(10^{2021}-1) = 1 - g(10^{2021}-1) = 1 - h(10^{2021}) = -2^{10105} + 1$, yielding our answer of 10104.