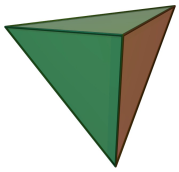
PUMaC 2020* Power Round

Spring 2021



Rules and Reminders

1. Your solutions should be turned in by **12PM Thursday, March 25th, EDT**. You will submit the solutions through Gradescope. The instructions describing how to log into Gradescope will be sent to the coaches. The deadline for submission is clearly visible on the Gradescope site once you enroll in the course.

   Please make sure you submit you work in time. **No late submissions will be accepted.** Please do not submit your work using email or in any other way. If you have questions about Gradescope, please post them on Piazza.

   You may either typeset the solutions in LaTeX or write them by hand. We strongly encourage you to typeset the solutions. This way, the proofs end up being more clear and the chances are you will not lose points there. Moreover, you might want to use some of the LaTeX resources listen in point 2.

   In case your solutions are handwritten, the cover sheet (the last page of this document) should be the first page of your submission. In case you typeset your solutions, please take a look at the Solutions Template we posted and make sure to make the cover sheet the first page of your submission.

   Each page should have on it the **team number** (not team name) and **problem number**. This number can be found by logging in to the coach portal and selecting the corresponding team. Solutions to problems may span multiple pages, but include them in continuing order of proof.

2. You are encouraged, but not required, to use LaTeX to write your solutions. If you submit your power round electronically, **may submit several times, but only your final submission will be graded** (moreover, you may not submit any work after the deadline). The last version of the power round solutions that we receive from your team will be graded. Moreover, **you must submit a PDF**. No other file type will be graded.

3. Do not include identifying information aside from your team number in your solutions.
4. Please collate the solutions in order in your submission. Each problem should start on a new page (there is a point deduction for not following this formatting).
5. On any problem, you may use without proof any result that is stated earlier in the test, as well as any problem from earlier in the test, even if it is a problem that your team has not solved. These are the **only** results you may use. In particular, to solve a problem, you may not cite the subsequent ones. You may not cite parts of your proof of other problems: if you wish to use a lemma in multiple problems, please reproduce it in each one.
6. When a problem asks you to "find", "find with proof," "show," "prove," "demonstrate," or "ascertain" a result, a formal proof is expected, in which you justify each step you take, either by using a method from earlier or by *proving* that *everything* you do is correct. When a problem instead uses the word "explain," an informal explanation suffices. When a problem instead uses the word "sketch" or "draw" a clearly marked diagram is expected.
7. All problems are numbered as "Problem x.y" where x is the section number and y is the the number of the problem within this section. Each problem's point distribution can be found in the cover sheet.
8. **You may NOT use any references, such as books or electronic resources, unless otherwise specified. You may NOT use computer programs, calculators, or any other computational aids.**
9. Teams whose members use English as a foreign language may use dictionaries for reference.
10. **Communication with humans outside your team of 8 students about the content of these problems is prohibited.**
11. There are two places where you may ask questions about the test. The first is Piazza. Please ask your coach for instructions to access our Piazza forum. On Piazza, you may ask any question **so long as it does not give away any part of your solution to any problem**. If you ask a question on Piazza, all other teams will be able to see it. If such a question reveals all or part of your solution to a power round question, your team's power round score will be penalized severely. For any questions you have that might reveal part of your solution, or if you are not sure if your question is appropriate for Piazza, please email us at pumacpowerround2020@gmail.com. We will email coaches with important clarifications that are posted on Piazza.

Introduction and Advice

This year's power round is about **polyhedra**. We will study various kinds of lines and segments on poolyhedra, combining their geometric and combinatorial properties. The questions will be motivated by extremal situations, such as making some paths as short as possible.

The power round is structured such that it will walk you through proofs of some of the important theorems, by giving you hints and problems along the way.

Here is some further advice with regard to the Power Round:

- **Read the text of every problem!** Many important ideas are included in problems and may be referenced later on. In addition, some of the theorems you are asked to prove are useful or even necessary for later problems.
- **Make sure you understand the definitions**. A lot of the definitions are not easy to grasp; don't worry if it takes you a while to fully understand them. If you don't, then you will not be able to do the problems. Feel free to ask clarifying questions about the definitions on Piazza (or email us).
- **Don't make stuff up**: on problems that ask for proofs, you will receive more points if you demonstrate legitimate and correct intuition than if you fabricate something that *looks* rigorous just for the sake of having "rigor."
- **Check Piazza often!** Clarifications will be posted there, and if you have a question it is possible that it has already been asked and answered in a Piazza thread (and if not, you can ask it, assuming it does not reveal any part of your solution to a question). **If in doubt about whether a question is appropriate for Piazza, please email us at pumacpowerround2020@gmail.com.**
- **Don't cheat**: as stated in Rules and Reminders, you may **NOT** use any references such as books or electronic resources. If you do cheat, you will be disqualified and banned from PUMaC, your school may be disqualified, and relevant external institutions may be notified of any misconduct.

Good luck, and have fun!

– Daniel Carter, Igor Medvedev, Aleksa Milojevic, Alan Yan

We would like to acknowledge and thank many individuals and organizations for their support; without their help, this Power Round (and the entire competition) could not exist. Please refer to the solutions of the power round for full acknowledgments and references.

Contents

Notation

- $\forall$: for all. *Ex.:* $\forall x \in \{1,2,3\}$ *means "for all* x *in the set* $\{1,2,3\}$*"*
- $A \subset B$: proper subset. *Ex.:* $\{1,2\} \subset \{1,2,3\}$, *but* $\{1,2\} \not\subset \{1,2\}$
- $A \subseteq B$: subset, possibly improper. *ex.:* $\{1\},\{1,2\} \subseteq \{1,2\}$
- $f : x \mapsto y$: f maps x to y. *Ex.: if* $f(n) = n-3$ *then* $f : 20 \mapsto 17$ *and* $f : n \mapsto n-3$ *are both true.*
- $\{x \in S : C(x)\}$: the set of all x in the set S satisfying the condition $C(x)$. *Ex.:* $\{n \in \mathbb{N} : \sqrt{n} \in \mathbb{N}\}$ *is the set of perfect squares.*
- $\mathbb{N}$: the natural numbers, $\{1,2,3,\dots\}$.
- $[n] = \{1,2,3,...,n\}$.
- $\mathbb{Z}$: the integers.
- $\mathbb{R}$: the real numbers.
- $|S|$: the cardinality of set S.

1 Playing Billiard

Alex loves playing billiard. Recently, he learned that the billiard balls bounce off the walls of the billiard table at the same angles they come in. Further, when a ball hits a corner of the table it may chose to bounce off any line between the two lines incident to that corner. Alex is a good geometer and has precisely defined this billiard game on an arbitrary convex polygon.

Definition 1.A. A broken line $A_1A_2\dots A_n$ is a union of line segments $A_1A_2, A_2A_3, \dots A_{n-1}A_n$. We sat a broken line is closed if $A_1 = A_n$ and we call $A_1, \dots, A_n$ the breakpoints of this broken line.

Definition 1.B. Let P be a convex polygon. A broken line $A_1A_2...A_n$ is called a *billiard trajectory* on P if:

- The points A_i lie on the boundary of P, for $i = 1, ..., n$.
- If A_i lies in the interior of the edge XY of P, we have $\angle A_{i-1}A_iX = \angle YA_iA_{i+1}$.
- If A_i is a vertex of P, adjacent to the vertices X and Y of P, then

$$|\angle XA_iA_{i-1} - \angle A_{i+1}A_iY| + \angle XA_iY \leq 180^\circ$$

Now, Alex is interested what kinds of trajectories he can find on a polygon, and has one more definition.

Definition 1.C. Let $A_1A_2...A_{n+1}$ be a billiard trajectory on a polygon P. If $A_1 = A_n$ and $A_2 = A_{n+1}$, then $A_1...A_n$ is called a *closed trajectory*. For a closed trajectory C consisting of n line segments, we define its order as $|C| = n$.

Clearly, if there is at least one closed trajectory C on a polygon P, there is infinitely many of them, obtained by walking several times over C. Trajectories obtained by repeating several times the same basic closed trajectory C are called *powers* of C. Trajectories that cannot be obtained as powers are called *prime*.

Theorem 1.I. For any convex polygon P there exist infinitely many prime closed billiard trajectories on P.

We will prove this theorem in several steps. The main idea is to take a longest closed broken line L with vertices on the boundary of P and at most n vertices, for some wisely chosen n.[1] The following three problems will show that this line satisfies the conditions of the theorem 1.I.

Problem 1.1. Prove that L has exactly n vertices.

Problem 1.2. Prove that L is indeed a closed billiard trajectory, in the sense of definitions 1.B and 1.C.

[1] Although it may not be utterly obvious that such a maximal broken line exists, this follows by a simple compactness argument. However, as this is not our topic here, you may assume without proof that such a maximal broken line exists.

Problem 1.3. Complete the proof of theorem 1.I.

A similar idea can be applied to prove another result of the same flavor.

Problem 1.4. Prove that, given any two points A, B on the boundary of a convex polygon P, there exists infinitely many billiard trajectories starting at A and ending at B.

2 Introduction to ant-paths

Alan the Ant lives on the surface of a three-dimensional polyhedron $\mathcal{P}$. His life consists of moving along this surface, in a specific manner. Every day, Alan chooses two points X and Y on that surface, positions himself at X and tries to get as fast as possible to Y. To do that, he needs to find a shortest path between X and Y on the surface of $\mathcal{P}$.

To formalize these concepts, we have the following definitions. Although these definition may seem cumbersome at first, they are only rephrasing the intuition we have about polyhedra and their surfaces in mathematical terms. The first definition gives us the concept of the polyhedron:

Definition 2.A. A *half-space* is a region on one side of the plane. Half-spaces can be *closed* or *open*, depending on whether they contain the bounding plane or not. A set of points in $\mathbb{R}^3$ is said to be *bounded* if there is a big enough ball containing it. A *polyhedron* is a bounded intersection of several closed half-spaces.

The following definition will formalize the concept of the surface of that polyhedron.

Definition 2.B. Let $\mathcal{P}$ be a polyhedron and let H be a plane which does not intersect the interior of $\mathcal{P}$. If the intersection of H and $\mathcal{P}$ is a point, that point is called a *vertex* of $\mathcal{P}$. If $H \cap \mathcal{P}$ is a line segment, that line segment is called an *edge* of $\mathcal{P}$, while if $H \cap \mathcal{P}$ is a plane polygon, it is called a *face* of $\mathcal{P}$. The set of vertices of $\mathcal{P}$ is denoted by $V(\mathcal{P})$. The *surface* of $\mathcal{P}$ is the union of all its faces. We denote the surface of $\mathcal{P}$ by $S(\mathcal{P})$.

Finally, the last definition explains what a path on the surface is and how we measure its length.

Definition 2.C. Let X and Y be two points on the surface of a polyhedron $\mathcal{P}$. A *path* between X and Y on the surface $S(\mathcal{P})$ is a continuous curve $C \subset S(\mathcal{P})$, having one end in X and the other end in Y. For any such curve C, we can pick several points along it, say $X = T_0, T_1, T_2, ..., T_k = Y$ in that order along C, such that the segments of C between T_i and T_{i+1} lie on only one face. Then, as the segments T_i and T_{i+1}, we can measure their length and denote it by $l(T_iT_{i+1})$. Now, we can define the *length* of C as $l(C) = l(T_0T_1) + l(T_1T_2) + \cdots + l(T_{k-1}T_k)$. A *shortest path* between X and Y is the path between X and Y with the minimum length.

It turns out that shortest paths on a surface of a polyhedron are exceedingly interesting, as the following properties show:

Problem 2.1. Let C be the shortest path between points X and Y on the surface of polyhedron $\mathcal{P}$. Prove that the interior of C does not contain any vertices of $\mathcal{P}$. In other words, C contains no vertices of $\mathcal{P}$ except maybe X and Y.

Hint: Argue by contradiction. Assume there is a shortest path C between X and Y going through the vertex V and use it to construct a path from X to Y shorter than C to get a contradiction.

Problem 2.2. Let C be the shortest path between points X and Y on the surface of polyhedron $\mathcal{P}$. Given an edge AB of $\mathcal{P}$, show that C intersects AB only in isolated points. In other words, show that C does not contain a subsegment of AB of nonzero length. Moreover, show that whenever C intersects AB, it changes the face (i.e. C uses the edge AB only when it goes from one face to the other).

Hint: Argue by contradiction, as before. Assume that some forbidden structure exists, and use it to construct a shorter path.

Problem 2.3. Let C be the shortest path between points X and Y on the surface of polyhedron $\mathcal{P}$. Given an edge AB of $\mathcal{P}$ we pick a subsegment KL of C which intersects AB at exactly one point, T. Show that $\angle KTA = \angle BTL$ (the angles are measured in their respective planes).

The above nice properties motivate the following definition, which generalizes the concept of shortest paths, while maintaining all its useful properties.

Definition 2.D. Let $\mathcal{P}$ be a polyhedron and $S(\mathcal{P})$ its surface. An *ant-path* on $S(\mathcal{P})$ is a piecewise linear curve C which satisfies the following: for every point P in the interior, there is a subsegment X_PY_P of C which contains P, and on which C agrees with the shortest path between X_P and Y_P.

In other words, an ant-path is a path which looks like a shortest path when zoomed in enough. As we shall see in the next problem, the properties that we proved for the shortest paths above are enough to guarantee something is an ant path.

Problem 2.4. Let C be a piecewise linear curve on the surface $S(\mathcal{P})$ of the polyhedron, with breaking points $T_1, T_2, ..., T_k$. Prove that C is an ant-path if and only if it satisfies the following three properties:

- All of the breaking points $T_1, T_2, ..., T_k$ lie on the edges of $\mathcal{P}$,
- C contains no vertices in its interior,
- If the breaking point T_i is contained in a short subsegment KL (K and L being on the different faces of $\mathcal{P}$) of C, and on the edge AB of $\mathcal{P}$, then $\angle KT_iA = \angle BT_iL$.

Remark: Formally, to solve this problem you need to prove two directions. However, you will find one of them easy due to the work we already did.

Of special interest are ant-paths that do not have beginnings or ends: the closed ant-paths.

Definition 2.E. A *closed* ant-path is a ant-path C whose beginning and end coincide. If this point is denoted by P, C needs to satisfy: there is a subsegment X_PY_P of C which contains P, and on which C agrees with the shortest path between X_P and Y_P. A closed ant-path is *simple* if it has no self-intersections.

Interestingly enough, the simple closed ant-paths are relatively rare. The following problem shows that generic tetrahedrons almost never contains simple closed ant-paths.

Problem 2.5. Let $A_1A_2A_3A_4$ be a tetrahedron, and let θ_i denote the sums of angles at the vertex A_i (in other words, $\theta_1 = \angle A_2A_1A_3 + \angle A_2A_1A_4 + \angle A_4A_1A_3$, and similarly for the other indices). If $\theta_i + \theta_j \neq 2\pi$ for all $i \neq j \in [4]$, then there is no simple closed ant-path with 3 or 4 segments on $S(A_1A_2A_3A_4)$.

Hint: Try to draw such a path and show it cannot exist due to angle constraints given by the problem 2.4.

The above result can be generalized to many other polyhedra. In some sense, almost no generic polyhedra have simple closed ant paths on their surfaces. However, in order to prove that, we will need to develop a stronger machinery.

3 More about polyhedra

We will now recall some of the well-known properties of the polyhedrons, and also introduce some on the new ones. One of the most famous results describing 3-dimensional polyhedra is the celebrated Euler's formula:

Problem 3.1. Let $\mathcal{P}$ be a polyhedron having V vertices, E edges and F faces. The following formula relates these three quantities: $V - E + F = 2$.

Hint: Form a planar graph out of this polyhedron and sum up its angles.

As we aim to generalize the result of problem 2.5, we will follow a similar path, and thus define the following quantity:

Definition 3.A. For a vertex v of the polyhedron $\mathcal{P}$, we define its *pointiness* as $p(v) = 2\pi - \sum_i \alpha_i$, where α_i are the face angles of $\mathcal{P}$ at the vertex v.

Although the following definition may seem slightly arbitrary at first, the following claim shows that it is actually useful:

Problem 3.2. For a polyhedron $\mathcal{P}$, we have $\sum_{v \in V(\mathcal{P})} p(v) = 4\pi$.

The preceding problem reminds us of two dimensional case, which we can perhaps solve first in order to build intuition.

Problem 3.3. Let $\mathcal{M}$ be a plane polygon, with vertices $X_1, ..., X_k$. For every vertex X_i, we can define the *outside angle* at X_i as π minus the angle of $\mathcal{M}$ at X_i. Then, the sum of outside angles is 2π.

After having learned how pointiness of vertices behave, we can link that behaviour with ant-paths.

Problem 3.4. Let C be a simple closed ant-path on the surface of $\mathcal{P}$. It divides the surface $S(\mathcal{P})$, and the vertices of $V(\mathcal{P})$ consequently, into two parts. The sum of pointiness of vertices in each part is equal to 2π.

The previous problem now gives an even simpler solution to problem 2.5. As there is no way to find two vertices with pointiness adding up to 2π (by the constraints of the problem), there are no simple closed ant-paths on the surface of the tetrahedron. Furthermore, the previous claim gives us a very strong tool when showing that a given polyhedron has no simple closed ant-paths - it is enough to check that no subset of its vertices has the sum of pointiness equal to 2π. In some sense, this means that the most of polyhedra do not have simple closed ant-paths on their surfaces (making these concepts more precise would be out of scope of this power round).

The inverse of the previous theorem does not hold, as we will show in the following problem.

Problem 3.5. Explicitly construct a polyhedron $\mathcal{P}$ with no simple closed ant-paths such that there is a set of its vertices $V_0 \subset V(\mathcal{P})$ in which $\sum_{v \in V_0} p(v) = 2\pi$.

There is one more, somewhat surprising result, concerning different ant-paths on polyhedrons:

Problem 3.6. Let $\mathcal{P}$ be a polyhedron with two different simple closed ant-paths C_1 and C_2 which do not intersect. Then, these ant-paths have the same length.

4 Ant-paths on tetrahedra

In this section we will examine the simple closed ant-paths on the surfaces of various tetrahedra. We start off by examining and classifying the ant-paths on the regular tetrahedron.

We are interested in a question of the form: what are all of the possible lengths of ant-paths on the surface of the regular tetrahedron. First, we will notice that a tetrahedron has a very special property that makes it even simpler than other regular polyhedra.

Problem 4.1. Let $ABCD$ be a regular tetrahedron, assume the face ABC is horizontal, and denote this plane by α. Prove that it is possible to roll the given tetrahedron on this plane such that the faces of the tetrahedron form a triangular tiling of the plane. Moreover, prove that it is possible to assign letters a, b, c, d to the vertices of this tiling such that the vertex A of the tetrahedron always lands on the vertex of the tiling marked by a, and similarly for the vertices B, C, D.

Having constructed this useful rolling of the tetrahedron on the plane, we can start dealing with ant-paths. Assume the $S(ABCD)$ contains an ant-path $\mathcal{C}$. Roll the tetrahedron $ABCD$ along this ant-path, until we come back where we started on the ant-path. In the plane, this ant-path will have the following form: its endpoints will be on the edges of the tiling marked by the same letters, and oriented the same way (e.g. both endpoints will be on the edges ab of the tiling, and a is left of b on both edges). Moreover, as long as such a plane segment does not contain any vertices of the tiling, it will be possible to uniquely bring it back on the surface of $ABCD$.

Problem 4.2. Prove that all closed ant-paths on the surface of $ABCD$ are simple. (You may assume that the ant-paths do not repeat themselves several times.)

Hint: How are the cells of the tiling corresponding to the same face oriented?

Now, we are able to produce an answer to our starting question:

Problem 4.3. Assume tetrahedron $ABCD$ has unit edge length. Find all possible lengths of closed ant-paths.

Now, having solved the main question in case of regular tetrahedron, we will broaden our focus. There is one specific family of tetrahedra, called *equihedral* tetrahedra which behave very nicely with respect to ant-paths. These tetrahedrons are defined as follows:

Definition 4.A. A tetrahedron $ABCD$ is called *equihedral* if its four faces ABC, ADC, ADB, and CDB are all congruent.

A useful way to visualize equihedral tetrahedra is as the face diagonals of a rectangular prism:

Problem 4.4. Prove that if $ABCD$ is an equihedral tetrahedron, there exists a rectangular prism where four of its eight vertices are A, B, C, and D, and the six edges of $ABCD$ are diagonals of the six faces of the prism.

There are many other statements which are equivalent to the given defintion of a tetrahedron being equihedral:

Problem 4.5. Let Δ be a tetrahedron with vertices $ABCD$. Prove that the following statements are equivalent:

- Δ is equihedral in the sense of Definition 4.A.
- The perimeters of all faces ABC, ABD, ACD, and BCD are equal.
- The pointiness of all vertices are equal, i.e. $p(A) = p(B) = p(C) = p(D)$.
- The dihedral angles at the opposite edges are equal. In other words, the angle between planes ABC and BCD is equal to the angle between the planes ACD and ABD (and similarly for other pairs of planes).
- The solid angles at each vertex have the same measure.

Remark: The solid angle at the vertex V of a tetrahedron can be defined as follows: if S is a sphere of radius r around V, r being small enough that this sphere does not intersect other edges of the tetrahedron except the ones incident to V, then the solid angle at V is the ratio of the area of S contained in V to the square of the radius.

Equihedral tetrahedra turn out to be exceedingly interesting when discussing ant-paths. The following problems show us several examples of this correlation.

Problem 4.6. Prove that there are three pairwise intersecting simple closed ant-paths on a tetrahedron if the tetrahedron is equihedral.

Problem 4.7. Two closed ant-paths on the surface of the tetrahedron are called *similar* if they intersect the edges of the tetrahedron in the exact same order. Prove that a tetrahedron has infinitely many non-similar closed ant-paths if and only if it is equihedral.

5 Ant-paths on cubes

Having thoroughly examined the case of both regular and equihedral tetrahedra, we turn to the case of the cube. More precisely, we are interested in which are the possible lengths of ant-paths on the surface of a cube. Unfortunately, when dealing with the cube, there is no statement analogous to problem 4.1. Therefore, when extending the claims about tetrahedra to cubes, we need to have a different approach when unrolling a cube onto the plane.

Problem 5.1. Let $ABCDA'B'C'D'$ be a cube and let α be the plane supporting the side $ABCD$. Rolling the cube over its edges onto α will give the integer-point lattice in the plane. Prove that it is not possible to label the vertices of this lattice by $a, b, c, d, a', b', c', d'$ so that the vertex A of the cube always lands at the point label a, vertex B at b, etc.

The last problem means that we will have to label the vertices on the plane depending on the ant-path we are examining at the moment. More precisely, to each closed ant-path, we will associate a straight planar segment of the same length. Then, we will examine which planar segments can be obtained from ant-paths, which will lead us towards a better understanding of ant-paths on the surface of the cube and of their lengths.

To construct a straight planar segment, we assume we have an ant-path $\mathcal{C}$ that intersects the edge AB and goes into the face $ABCD$ afterwards. Moreover, assume $X \in \mathcal{C} \cap AB$ and $XA = d$. For the sake of simplicity we place the face $ABCD$ onto the unit square on the plane, and let X have coordinates $(d, 0)$. Then, we unroll the cube along $\mathcal{C}$. At some point, after the whole ant-path is unrolled, the point X will come back on the plane, thus marking the point X'.

Problem 5.2. Prove that the point X' will have the coordinates of the form $(d + m, n)$, for some positive integers $m \in \mathbb{Z}, n \in \mathbb{Z}_{>0}$. Further, prove that the length of the planar line segment XX' is equal to the length of $\mathcal{C}$. Here, you may assume $\mathcal{C}$ is simple if that helps.

However, not every segment going from $(d, 0)$ to $(d + m, n)$ corresponds to an ant-path.

Problem 5.3. Give an example showing the previous claim (i.e. find some d, m, n such that the segment from $(d, 0)$ to $(d + m, n)$ cannot be produced from the ant-path using the above procedure).

Therefore, our goal is to find which segments from $X = (d, 0)$ to $X' = (d + m, n)$ correspond to actual closed ant-paths on the cube. There is one easy way to determine this for any d, m, n. Start by labeling the vertices of the unit square by a, b, c, d and by unrolling the cube over the segment XX'. When a vertex V of the cube falls onto the plane, label the point by v ($V \in \{A, B, C, D, A', B', C', D'\}$). If X' ends up on the horizontal edge whose left label is a and whose right label is b (the left label corresponding to the point (m, n), and the right on to the point $(m + 1, n)$), then the cube have made a full rotation and we have an ant-path.

Note that the segment XX' contains no points with integer coordinates. From now on, whenever we consider segments of this type, we will exclude the segments containing the integer points.

In general, given some fixed m, n, and $d \in [0,1]$, the labels of the points (m,n) and $(m+1,n)$ could depend on d. However, we have the following claim that eliminates this possibility in case the points (m,n) and $(m+1,n)$ were labeled a and b, as the following problem suggests.

Problem 5.4. If for given $m, n \in \mathbb{Z}_+$, the points (m,n) and $(m+1,n)$ get labels a, b for some d under the described procedure, they get the same labels for all but finitely many $d \in [0,1]$.

The last problem tells us the fact whether the triple (m,n,d) corresponds to an ant-path does not really depend on d - if (m,n,d) comes from an ant-path for one value of d, it does so for almost any other value of d. Thus, one can define the pair (m,n) to be good if there is some d such that (m,n,d) corresponds to an ant-path. Moreover, as we have it is easy to see that m and n are enough to determine the length of the ant-path $\mathcal{C}$, because the length of the segment XX' is $XX' = \sqrt{m^2+n^2}$.

This property helps us determine the lengths of all the ant-paths on a cube. However, before doing that, we will classify the simple - non-intersecting ant paths on a cube. To this end, we have two lemmas that provide a great reduction in terms of various m and n we have to consider.

Problem 5.5. Let $\mathcal{C}$ be a closed ant-path on the surface of the cube which has a self-intersection at a point A. Prove that the ant-path is perpendicular to itself at A.

Problem 5.6. Let $\mathcal{C}$ be a closed ant-path on the surface of the cube which has an unfolding on the plane from $(d,0)$ to $(m+d,n)$. Prove that if $m+n \geq 7$, the ant-path has a self-intersection.

Using the last problem, we can determine the lengths of all non-intersecting ant-paths on a cube.

Problem 5.7. Find the lengths of all non self-intersecting closed ant-paths on the surface of a cube. For each length you find, sketch an ant-path of that length on the surface of the cube.

Further, we will prove that we can find an ant-path corresponding to almost any direction in the plane.

Problem 5.8. Let m, n be two coprime non-negative integers (not both being zero). Then, there exists a unique integer $k \leq 4$ for which the pair (km, kn) corresponds to an ant-path (that does not repeat itself).

Hint: If (m,n) is not a good pair by itself, what are the labels of the points (m,n) and $(m+1,n)$. Can you make any inference about the labels of $(2m,2n)$ and $(2m,2n+1)$ based on this?

The statement of the last problem seems incomplete: how to determine k based on m,n? The following asks for any progress towards determining which k corresponds to which pairs m, n:

Problem 5.9. Prove that k always comes from $\{2,3,4\}$ in the last problem. Determine which values can k take depending on the parity of m, n.

Hint: Although there is a decisive answer to this question, this problem is intended to be open ended. Whatever conclusions you might have about how to determine k, make sure to write them down.

Finally, to outline the usefulness of this approach, we pose a computational question:

Problem 5.10. What are the five shortest lengths of closed, non-repeating ant-paths on the surface of the cube?

Team Number: ______________________________

PUMaC 2020* Power Round Cover Sheet

Remember that this sheet comes first in your stapled solutions. You should submit solutions for the problems in increasing order. Write on one side of the page only. The start of a solution to a problem should start on a new page. Please mark which questions for which you submitted a solution to help us keep track of your solutions.

Problem Number	Points	Attempted?
1.1	10	
1.2	20	
1.3	20	
1.4	15	
2.1	20	
2.2	20	
2.3	20	
2.4	20	
2.5	30	
3.1	10	
3.2	20	
3.3	5	
3.4	35	
3.5	25	
3.6	30	
4.1	10	
4.2	20	
4.3	30	
4.4	20	
4.5	60	
4.6	50	
4.7	50	
5.1	10	
5.2	20	
5.3	10	
5.4	40	
5.5	20	
5.6	30	
5.7	30	
5.8	30	
5.9	60	
5.10	20	