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Algebra A Solutions

1. Compute the sum of all real numbers x which satisfy the following equation

$$\frac{8^x - 19 \cdot 4^x}{16 - 25 \cdot 2^x} = 2.$$

Proposed by: Nancy Xu

Answer: $\boxed{5}$

Let $y = 2^x$. Then the equation becomes $\frac{y^3 - 19y^2}{16 - 25y} = 2$, which gives us $y^3 - 19y^2 + 50y - 32 = 0$. All roots of this polynomial must divide 32, so by testing the divisors of 32 we find that $y^2 - 19y^2 + 50y - 32 = (y-1)(y-2)(y-16)$, so that $x = 0, 1$, or 4. Thus the desired sum is $0 + 1 + 4 = 5$.

2. For a bijective function $g : \mathbb{R} \to \mathbb{R}$, we say that a function $f : \mathbb{R} \to \mathbb{R}$ is its *superinverse* if it satisfies the following identity $(f \circ g)(x) = g^{-1}(x)$, where g^{-1} is the inverse of g. Given $g(x) = x^3 + 9x^2 + 27x + 81$ and f is its superinverse, find $|f(-289)|$.

Proposed by: Henry Erdman

Answer: $\boxed{7}$

Applying each side of the identity to g^{-1} gives $f \circ g \circ g^{-1} = g^{-1} \circ g^{-1}$. Noting that $g \circ g^{-1}$ is just the identity function, we have $f = g^{-1} \circ g^{-1}$. Computing from $g(x) = (x+3)^3 + 54$, we have $g^{-1}(x) = (x - 54)^{\frac{1}{3}} - 3$. Since $g^{-1}(-289) = -10$, and $g^{-1}(-10) = -7$, $|f(-289)| = 7$.

3. Let $f(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4$ and let $\zeta = e^{2\pi i/5} = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$. Find the value of the following expression:

$$f(\zeta)f(\zeta^2)f(\zeta^3)f(\zeta^4).$$

Proposed by: Michael Gintz

Answer: $\boxed{125}$

Write this as the product

$$\begin{aligned} f(x) &= \frac{(x^5 - 1) + (x^5 - x) + \ldots + (x^5 - x^4)}{x - 1} \\ &= \frac{5x^6 - 6x^5 + 1}{(x-1)^2}, \end{aligned}$$

which for these terms will be equal to $\frac{5}{(x-1)}$. Thus taking this for each of our four multiplicands, the denominator becomes $((1-\zeta)(1-\zeta^2)(1-\zeta^3)(1-\zeta^4) = 5$, so our answer is 125.

4. The roots of a monic cubic polynomial p are positive real numbers forming a geometric sequence. Suppose that the sum of the roots is equal to 10. Under these conditions, the largest possible value of $|p(-1)|$ can be written as $\frac{m}{n}$, where m, n are relatively prime integers. Find $m + n$.

Proposed by: Frank Lu

Answer: $\boxed{2224}$

Because the cubic has roots that form a geometric sequence, we may write the cubic in the form $p(x) = (x - d)(x - dr)(x - \frac{d}{r})$, where d, r are positive numbers. We are given that

$d + dr + \frac{d}{r} = 10$, and we want to find the maximum of $|p(-1)|$. In other words, we want to find the maximum value of $1 + d + dr + \frac{d}{r} + d^2(1 + r + \frac{1}{r}) + d^3$.

But given that $d + dr + \frac{d}{r} = 10$, notice that we can write this as $d^3 + 10d + 10 + 1 = d^3 + 10d + 11$. Notice in fact that we can write $d^3 + 10d = (d^2 + 10)d$, both of which are increasing with d. Therefore, it suffices for us to find the largest possible value of d, as this will then give us the largest possible value of $|p(-1)|$.

However, given that $d + dr + \frac{d}{r} = 10$, we see that d is maximized when $1 + r + \frac{1}{r}$ is minimized. But $r + \frac{1}{r} \geq 2$; thus, we have that $d = \frac{10}{3}$ maximizes $p(-1)$. Substituting this value in, we get our maximal value for $|p(-1)|$ as $\frac{1000}{27} + \frac{100}{3} + 11 = \frac{1000+900+297}{27} = \frac{2197}{27}$. Our sum for $m + n$ is thus 2224.

5. The sum $\sum_{j=1}^{2021} |\sin \frac{2\pi j}{2021}|$ can be written as $\tan(\frac{c\pi}{d})$ for some relatively prime positive integers c, d, such that $2 * c < d$. Find the value of $c + d$.

   Proposed by: Frank Lu

   Answer: $\boxed{3031}$

   We know that the terms are positive for $j = 1, 2, \ldots, 1010$, and that they are negative for $j = 1011, 1012, \ldots, 2020$, and 0 for $j = 2021$. Furthermore, notice that $\sin \frac{2\pi j}{2021} = -\sin \frac{2\pi(2021-j)}{2021}$, so therefore this sum can be written as $2 \sum_{j=1}^{1010} \sin \frac{2\pi j}{2021}$.

   From here, we can think about this as the imaginary part of the sum of exponentials $2 \sum_{j=1}^{1010} e^{\frac{2i\pi j}{2021}}$. From the formula for a geometric series, we may write this as $2\frac{e^{\frac{2022i\pi}{2021}} - e^{\frac{2i\pi}{2021}}}{e^{\frac{2i\pi}{2021}} - 1}$. But recall that $e^{i\pi} = -1$, meaning that this is just equal to $2\frac{-e^{\frac{i\pi}{2021}} - e^{\frac{2i\pi}{2021}}}{e^{\frac{2i\pi}{2021}} - 1}$. From here, we may further simplify this as $2\frac{-e^{\frac{i\pi}{2021}}}{e^{\frac{i\pi}{2021}} - 1}$.

   Finally, observe that if we add 1 to this, we will not change the imaginary part of this value. But adding one yields us with the fraction $\frac{-e^{\frac{i\pi}{2021}} + 1}{e^{\frac{i\pi}{2021}} - 1} = \cot(\frac{\pi}{4042}) = \tan(\frac{\pi}{2} - \frac{\pi}{4042}) = \tan(\frac{1010\pi}{2021})$. Our answer is thus $1010 + 2021 = 3031$.

6. Let f be a polynomial. We say that a complex number p is a *double attractor* if there exists a polynomial $h(x)$ so that $f(x) - f(p) = h(x)(x - p)^2$ for all $x \in \mathbb{R}$. Now, consider the polynomial

   $$f(x) = 12x^5 - 15x^4 - 40x^3 + 540x^2 - 2160x + 1,$$

   and suppose that it's double attractors are $a_1, a_2, \ldots, a_n$. If the sum $\sum_{i=1}^{n} |a_i|$ can be written as $\sqrt{a} + \sqrt{b}$, where a, b are positive integers, find $a + b$.

   Proposed by: Frank Lu

   Answer: $\boxed{49}$

   Let $f(x) = 12x^5 - 15x^4 - 40x^3 + 540x^2 - 2160x + 1$. Notice that that $f(x) - f(p) = h(x)(x - p)^2$ if and only if $f(x + p) - f(p) = h(x + p)x^2$. But notice that $f(x + p) - f(p)$ is also a polynomial in x that equals 0 when $x = 0$. This is divisible by x^2 if and only if the coefficient of x is 0. But notice that, for p to satisfy this, we see that the x coefficient of $f(x + p) - f(p)$ is the same as that of $f(x + p)$, which is just $60p^4 - 60p^3 - 120p^2 + 1080p - 2160$. We will now figure out when this is 0. We re-write this, factoring out the 60, as $p^4 - p^3 - 2p^2 + 18p - 36$. Notice that this has integer solutions $2, -3$, and factoring yields $(p-2)(p+3)(p^2-2p+6)$. This yields two more solutions $1+i\sqrt{5}, 1-i\sqrt{5}$. Our final answer is thus just $2+3+2\sqrt{6} = 5+2\sqrt{6} = \sqrt{25}+\sqrt{24}$. Hence, we see that yields 49, as desired.

7. Consider the following expression

$$S = \log_2\left(\Big|\sum_{k=1}^{2019}\sum_{j=2}^{2020}\log_{2^{1/k}}(j)\log_{j^2}(\sin\frac{\pi k}{2020})\Big|\right).$$

Find the smallest integer n which is bigger than S (i.e. find $\lceil S \rceil$).

Proposed by: Frank Lu

Answer: $\boxed{31}$

We first write the inner expression as

$$\sum_{k=1}^{2019}\sum_{j=2}^{2020}\log_{2^{1/k}}(\sin\frac{\pi k}{2020})\log_{j^2}(j) = (\sum_{k=1}^{2019}\log_{2^{1/k}}(\sin\frac{\pi k}{2020}))(\sum_{j=2}^{2019}\log_{j^2}(j)).$$

This second term just evaluates to $\frac{2019}{2}$. For the first product, note that this is just $\sum_{k=1}^{2019} k\log_2(\sin\frac{\pi k}{2020})$, which we can rewrite as

$$\sum_{k=1}^{1010}(k+(2020-k))\log_2(\sin\frac{\pi k}{2020}) = 1010\sum_{k=1}^{2019}\log_2(\sin\frac{\pi k}{2020}).$$

Now, this is just $1010\log_2(\prod_{k=1}^{2019}\sin\frac{\pi k}{2020})$, so it suffices to find the product of these sines. Observe that $\prod_{k=1}^{2019}\sin\frac{\pi k}{2020} = \prod_{k=1}^{2019}\frac{(\omega^k-\omega^{-k})}{2i}$, where $\omega = e^{\frac{i\pi}{2020}}$. This in turn is just $\frac{w^{-1-2\ldots-2019}}{(2i)^{2019}}\prod_{k=1}^{2019}(w^{2k}-1)$. But this is then $-\frac{w^{-2019*1010}}{(2i)^{2019}}\prod_{k=1}^{2019}(1-w^{2k})$. We then evaluate this to equal $\frac{e^{\frac{-2019i\pi}{2}}}{2^{2019}i}\prod_{k=1}^{2019}(1-w^{2k})$. The terms within the product, however, just evaluate to 2020, as w^{2k} are the roots of $x^{2020}+\ldots+1$. Thus, we see that this equals $\frac{2020}{2^{2019}}$, implying that our answer for this inner logarithm is $|\frac{2019*1010}{2}\log_2\frac{2020}{2^{2019}}|$. To get $\log_2|\frac{2019*1010}{2}\log_2\frac{2020}{2^{2019}}|$, we write this as $\log_2(\frac{2019*1010}{2})+\log_2|\log_2 2020-2019|$. The second term we can bound between $\log_2 2008$ and $\log_2 2009$, both of which are larger than 10.5 as $1024\sqrt{2} < 1.5*1024$ and less than 11. Similarly, we see that $\log_2(\frac{2019*1010}{2})$ is less than 20 and larger than 19.5, since we have that $(1024)^2 = 1048576$, so $2^{19.5} < 1.5\cdot 2^{19} = 1.5*524288 < 10^6 < 2019*1010/2$, giving us an answer of 31.

8. Consider the sequence of Fibonacci numbers $F_0, F_1, F_2, \ldots$, given by $F_0 = F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. Define the sequence $x_0, x_1, x_2, \ldots$ by $x_0 = 1$ and $x_{k+1} = x_k^2 + F_{2^k}^2$ for $k \geq 0$. Define the sequence $y_0, y_1, y_2, \ldots$ by $y_0 = 1$ and $y_{k+1} = 2x_ky_k - y_k^2$ for $k \geq 0$. If

$$\sum_{k=0}^{\infty}\frac{1}{y_k} = \frac{a-\sqrt{b}}{c}$$

for positive integers a, b, c with $\gcd(a,c) = 1$, find $a+b+c$.

Proposed by: Sunay Joshi

Answer: $\boxed{14}$

Let $f(n) = F_n$. We claim that for all $k \geq 0$, we have $x_k = f(2^k+1)$ and $y_k = f(2^k)$. To see this, we proceed by induction on k. The base case is clear. Assume the result holds k. Then

$x_{k+1} = f(2^k+1)^2 + f(2^k)^2 = f(2^{k+1}+1)^2$, using the identity $f(2i+1) = f(i)^2 + f(i+1)^2$. Thus, $y_{k+1} = y_k(2x_k - y_k) = f(2^k)(f(2^k-1) + f(2^k+1)) = f(2^{k+1})$, using the identity $f(2i) = f(i)(f(i-1) + f(i+1))$. The induction is complete.

From the above, we must compute

$$\sum_{k=0}^{\infty} \frac{1}{F_{2^k}}.$$

However, this is simply the Millin series, with value $\frac{7-\sqrt{5}}{2}$. Thus our answer is $7+5+2 = 14$.