Algebra A

1. Compute the sum of all real numbers x which satisfy the following equation
$$\frac{8^x - 19 \cdot 4^x}{16 - 25 \cdot 2^x} = 2.$$

2. For a bijective function $g : \mathbb{R} \to \mathbb{R}$, we say that a function $f : \mathbb{R} \to \mathbb{R}$ is its *superinverse* if it satisfies the following identity $(f \circ g)(x) = g^{-1}(x)$, where g^{-1} is the inverse of g. Given $g(x) = x^3 + 9x^2 + 27x + 81$ and f is its superinverse, find $|f(-289)|$.

3. Let $f(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4$ and let $\zeta = e^{2\pi i/5} = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$. Find the value of the following expression:
$$f(\zeta)f(\zeta^2)f(\zeta^3)f(\zeta^4).$$

4. The roots of a monic cubic polynomial p are positive real numbers forming a geometric sequence. Suppose that the sum of the roots is equal to 10. Under these conditions, the largest possible value of $|p(-1)|$ can be written as $\frac{m}{n}$, where m, n are relatively prime integers. Find $m + n$.

5. Consider the sum
$$S = \sum_{j=1}^{2021} |\sin \frac{2\pi j}{2021}|.$$
The value of S can be written as $\tan(\frac{c\pi}{d})$ for some relatively prime positive integers c, d, satisfying $2c < d$. Find the value of $c + d$.

6. Let f be a polynomial. We say that a complex number p is a *double attractor* if there exists a polynomial $h(x)$ so that $f(x) - f(p) = h(x)(x-p)^2$ for all $x \in \mathbb{R}$. Now, consider the polynomial
$$f(x) = 12x^5 - 15x^4 - 40x^3 + 540x^2 - 2160x + 1,$$
and suppose that it's double attractors are $a_1, a_2, \ldots, a_n$. If the sum $\sum_{i=1}^{n} |a_i|$ can be written as $\sqrt{a} + \sqrt{b}$, where a, b are positive integers, find $a + b$.

7. Consider the following expression
$$S = \log_2 \Big(\Big| \sum_{k=1}^{2019} \sum_{j=2}^{2020} \log_{2^{1/k}}(j) \log_{j^2}(\sin \frac{\pi k}{2020}) \Big| \Big).$$
Find the smallest integer n which is bigger than S (i.e. find $\lceil S \rceil$).

8. Consider the sequence of Fibonacci numbers $F_0, F_1, F_2, \ldots$, given by $F_0 = F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. Define the sequence $x_0, x_1, x_2, \ldots$ by $x_0 = 1$ and $x_{k+1} = x_k^2 + F_{2^k}^2$ for $k \geq 0$. Define the sequence $y_0, y_1, y_2, \ldots$ by $y_0 = 1$ and $y_{k+1} = 2x_k y_k - y_k^2$ for $k \geq 0$. If
$$\sum_{k=0}^{\infty} \frac{1}{y_k} = \frac{a - \sqrt{b}}{c}$$
for positive integers a, b, c with $\gcd(a, c) = 1$, find $a + b + c$.