P U M $\therefore$ C

Combinatorics B Solutions

1. A nonempty word is called pronounceable if it alternates in vowels (A, E, I, O, U) and consonants (all other letters) and it has at least one vowel. How many pronounceable words can be formed using the letters P, U, M, A, C at most once each?

   Proposed by: Daniel Carter

   Answer: $\boxed{68}$

   The number of vowels and consonants to choose which can be arranged to form a pronounceable word are:

   - 1 vowel, 0 consonants: $2 \cdot 1 \cdot 1$ choices (2 choices of vowels, 1 of consonants, and 1 for order).
   - 1 V, 1 C: $2 \cdot 3 \cdot 2$ choices.
   - 1 V, 2 C: $2 \cdot 3 \cdot 2$ choices.
   - 2 V, 1 C: $1 \cdot 3 \cdot 2$ choices.
   - 2 V, 2 C: $1 \cdot 3 \cdot 8$ choices.
   - 2 V, 3 C: $1 \cdot 1 \cdot 12$ choices.

   The total is 68.

2. Neel and Roshan are going to the Newark Liberty International Airport to catch separate flights. Neel plans to arrive at some random time between 5:30 am and 6:30 am, while Roshan plans to arrive at some random time between 5:40 am and 6:40 am. The two want to meet, however briefly, before going through airport security. As such, they agree that each will wait for n minutes once he arrives at the airport before going through security. What is the smallest n they can select such that they meet with at least 50% probability? The answer will be of the form $a + b\sqrt{c}$ for integers a, b, and c, where c has no perfect square factor other than 1. Report $a + b + c$.

   Proposed by: Rishi Dange

   Answer: $\boxed{67}$

   Use geometric probability to see that the desired n will occur where $\frac{(50-n)^2}{2} + \frac{(70-n)^2}{2} = 0.5 \times 3600$. The larger solution obviously is not the correct one, leaving the smaller solution as the answer $(60 - 10\sqrt{17})$.

3. Select two diagonals at random from a regular octagon. What is the probability that the two diagonals intersect at a point strictly within the octagon? Express your answer as $a + b$, where the probability is $\frac{a}{b}$ and a and b are relatively prime positive integers.

   Proposed by: Rishi Dange

   Answer: $\boxed{26}$

   The answer is $\frac{7}{19}$. We use complementary counting.

   Split into three cases in which the diagonals don't intersect strictly within the octagon.

   Case 1: The diagonals intersect on the octagon. For each vertex, there are 5 possible diagonals from which we must choose 2 in this case. This gives us $8\binom{5}{2} = 80$ total ways.

   Case 2: The diagonals never intersect. Take a diagonal separated by 1 vertex. There are 2 other diagonals parallel to this one. There are 4 distinct possible rotations, giving a total of

$4\binom{3}{2} = 12$ possibilities. The only other way for the diagonals to not intersect is to take a diagonal separated by 2 vertices, in which case there is only 1 parallel choice and 4 rotations. Thus, we have a total of $12 + 4 = 16$ ways for this case.

Case 3: The diagonals intersect outside of the octagon. There are two subcases here. First, we can have two non-parallel diagonals that are each separated by 1 vertex. By rotation, there are 8 ways for this. Second, we can have one diagonal separated by 1 vertex and one separated by 2. For each possible diagonal separated by 1 vertex, there are 2 possibilities for the other diagonal. Thus, we have $8 \times 2 = 16$ possibilities here. That gives us a total of $8 + 16 = 24$ ways for this case.

The total number of ways to select diagonals that do not intersect strictly within the octagon is $80 + 16 + 24 = 120$. There are $\frac{8\times 5}{2} = 20$ diagonals, giving $\binom{20}{2} = 190$ total choices. Thus, by complementary counting, we arrive at a final answer of $1 - \frac{120}{190} = \frac{7}{19}$.

4. Eighteen people are standing in a (socially-distanced) line to enter a grocery store. Five people are wearing a black mask, 6 are wearing a gray mask, and 7 are wearing a white mask. Suppose that these 18 people got on line in a random order. The expected number of pairs of adjacent people wearing different-colored masks can be given by $\frac{a}{b}$. Compute $a + b$.

   Proposed by: Nancy Xu

   Answer: $\boxed{116}$

   For each pair of adjacent people, the probability that they are wearing different-colored masks is $\frac{5}{18} \cdot \frac{13}{17} + \frac{6}{18} \cdot \frac{12}{17} + \frac{7}{18} \cdot \frac{11}{17} = \frac{107}{153}$. There are 17 pairs of adjacent people, so the expected value is $\frac{107}{153} \cdot 17 = \frac{107}{9}$, and our answer is $107 + 9 = 116$.

5. Nelson is having his friend drop his unique bouncy ball from a 12 foot building, and Nelson will only catch the ball at the peak of its trajectory between bounces. On any given bounce, there is an 80% chance that the next peak occurs at $\frac{1}{3}$ the height of the previous peak and a 20% chance that the next peak occurs at 3 times the height of the previous peak (where the first peak is at 12 feet). If Nelson can only reach 4 feet into the air and will catch the ball as soon as possible, the probability that Nelson catches the ball after exactly 13 bounces is $2^a \times 3^b \times 5^c \times 7^d \times 11^e$ for integers a, b, c, d, and e. Find $|a| + |b| + |c| + |d| + |e|$.

   Proopposed by: Rishi Dange

   Answer: $\boxed{31}$

   Call an 80% bounce A and a 20% bounce B. It's easy to see that for 13 bounces to occur before ending, there must be 7 As and 6 Bs. Now we simply need to find the number of orderings of the As and Bs that are possible.

   The first bounce must be B and the last two must both be A, otherwise Nelson will catch the ball before 13 bounces. For the remaining 10 slots, there must be 5 As and 5 Bs such that within those 10, there is never an instance where the number of As exceeds the number of Bs by 2 or more (not including the bounce before those 10). That way, Nelson won't catch the ball before the 13 bounce total. This is a simple Catalan counting problem that gives 132 orderings, or $C(6)$.

   Thus, the final probability is $(\frac{4}{5})^7 \times (\frac{1}{5})^6 \times 132$, or $2^{16} \times 3^1 \times 5^{-13} \times 7^0 \times 11^1$.

6. There are n lilypads in a row labeled $1, 2, ..., n$ from left to right. Fareniss the Frog picks a lilypad at random to start on, and every second she jumps to an adjacent lilypad; if there are two such lilypads, she is twice as likely to jump to the right as to the left. After some finite number of seconds, there exists two lilypads A and B such that Fareniss is more than 1000 times as likely to be on A as she is to be on B. What is the minimal number of lilypads n such that this situation must occur?

Proposed by: Austen Mazenko

Answer: $\boxed{12}$

7. A Princeton slot machine has 100 pictures, each equally likely to occur. One is a picture of a tiger. Alice and Bob independently use the slot machine, and each repeatedly makes independent plays. Alice keeps playing until she sees a tiger, at which point she stops. Similarly, Bob keeps playing until he sees a tiger. Given that Bob plays twice as much as Alice, let the expected number of plays for Alice be $\frac{a}{b}$ with a, b relatively prime positive integers. Find the remainder when $a+b$ is divided by 1000.

Proposed by: Sunay Joshi

Answer: $\boxed{701}$

For simplicity, let $M = 100$. Let the random variables N_A and N_B denote the number of plays for Alice and Bob, respectively. We want to find the conditional expectation $E = E(N_A|N_B = 2N_A)$. By the Tower Law, we have

$$E = E(N_A|N_B = 2N_A) = \sum_{k=1}^{\infty} E(N_A|N_A = k, N_B = 2N_A)P(N_A = k|N_B = 2N_A) = \sum_{k=1}^{\infty} kP(N_A = k|N_B = 2N_A).$$

By definition,

$$P(N_A = k|N_B = 2N_A) = \frac{P(N_A = k, N_B = 2k)}{P(N_B = 2N_A)} = \frac{P(N_A = k, N_B = 2k)}{\sum_{\ell \geq 1} P(N_A = \ell, N_B = 2\ell)}.$$

As $P(N_A = k) = \left(\frac{M-1}{M}\right)^{k-1} \frac{1}{M}$ and $P(N_B = 2k) = \left(\frac{M-1}{M}\right)^{2k-1} \frac{1}{M}$, we find $P(N_A = k, N_B = 2k) = \left(\frac{M-1}{M}\right)^{3k} \frac{1}{(M-1)^2}$. Thus our denominator becomes

$$\sum_{\ell \geq 1} P(N_A = \ell, N_B = 2\ell) = \sum_{\ell \geq 1} \left(\frac{M-1}{M}\right)^{3\ell} \frac{1}{(M-1)^2}$$

$$= \frac{1}{(M-1)^2} \cdot \frac{\left(\frac{M-1}{M}\right)^3}{1 - \left(\frac{M-1}{M}\right)^3} = \frac{(M-1)^3}{M^3 - (M-1)^3} \frac{1}{(M-1)^2}.$$

Thus our conditional probability is given as

$$P(N_A = k|N_B = 2N_A) = \frac{\left(\frac{M-1}{M}\right)^{3k} \frac{1}{(M-1)^2}}{\frac{(M-1)^3}{M^3-(M-1)^3} \frac{1}{(M-1)^2}}$$

$$= \frac{M^3 - (M-1)^3}{(M-1)^3} \left(\frac{M-1}{M}\right)^{3k}.$$

Therefore our expected value reduces to

$$E = \sum_{k \geq 1} k \cdot \frac{M^3 - (M-1)^3}{(M-1)^3} \left(\frac{M-1}{M}\right)^{3k} = \frac{M^3 - (M-1)^3}{(M-1)^3} \sum_{k=1}^{\infty} kr^k,$$

where $r = \left(\frac{M-1}{M}\right)^3$. By differentiating the geometric series $\frac{1}{1-r} = 1 + r + r^2 + \ldots$ and multiplying by r, we see that $\sum_{k=1}^{\infty} kr^k = \frac{r}{(1-r)^2}$, hence the desired expectation is

$$E = \frac{M^3 - (M-1)^3}{(M-1)^3} \frac{r}{(1-r)^2} = \frac{M^3}{M^3 - (M-1)^3}.$$

For $M = 100$, the reduced fraction is $\frac{a}{b} = \frac{1000000}{29701}$. Thus $a + b \equiv 701 \pmod{1000}$, our answer.

8. Alice, Bob, and Carol are playing a game. Each turn, one of them says one of the 3 players' names, chosen from {Alice, Bob, Carol} uniformly at random. Alice goes first, Bob goes second, Carol goes third, and they repeat in that order. Let E be the expected number of names that are have been said when, for the first time, all 3 names have been said twice. If $E = \frac{m}{n}$ for relatively prime positive integers m and n, find $m + n$. (Include the last name to be said twice in your count.)

   Proposed by: Sunay Joshi

   Answer: $\boxed{383}$

   Let $E_{m,n}$ denote the expected number of *additional* names that must be said until all 3 names have been said twice, starting with m names said once, n names said at least twice, and $3 - m - n$ names said 0 times.

   We know that $E_{0,3} = 0$, and we wish to find $E = E_{0,0}$.

   We derive a recurrence relation for $E_{m,n}$. If a name is said for the first time on the next turn, then it contributes $\frac{3-m-n}{3}(E_{m+1,n}+1)$ to the expected value. Similarly, if one of the m names already said once is said again, we get a term of $\frac{m}{3}(E_{m-1,n+1} + 1)$. Finally, if one of the n names already said twice is said again, we get $\frac{n}{3}(E_{m,n} + 1)$. Adding these up, we find the recurrence

$$E_{m,n} = 1 + \frac{3-m-n}{3}E_{m+1,n} + \frac{m}{3}E_{m-1,n+1} + \frac{n}{3}E_{m,n},$$

   which simplifies to

$$\frac{3-n}{3}E_{m,n} = \frac{3-m-n}{3}E_{m+1,n} + \frac{m}{3}E_{m-1,n+1} + 1.$$

   Solving the resulting system of 9 equations, we find that $E = E_{0,0} = \frac{347}{36}$, and our answer is $m + n = 347 + 36 = 383$.