



Combinatorics A

1. Select two distinct diagonals at random from a regular octagon. What is the probability that the two diagonals intersect at a point strictly within the octagon? Express your answer as $a + b$, where the probability is $\frac{a}{b}$ and a and b are relatively prime positive integers.
2. Eighteen people are standing in a (socially-distanced) line to enter a grocery store. Five people are wearing a black mask, 6 are wearing a gray mask, and 7 are wearing a white mask. Suppose that these 18 people got on line in a random order. The expected number of pairs of adjacent people wearing different-colored masks can be given by $\frac{a}{b}$, where $\gcd(a, b) = 1$. Compute $a + b$.
3. Nelson is having his friend drop his unique bouncy ball from a 12 foot building, and Nelson will only catch the ball at the peak of its trajectory between bounces. On any given bounce, there is an 80% chance that the next peak occurs at $\frac{1}{3}$ the height of the previous peak and a 20% chance that the next peak occurs at 3 times the height of the previous peak (where the first peak is at 12 feet). If Nelson can only reach 4 feet into the air and will catch the ball as soon as possible, the probability that Nelson catches the ball after exactly 13 bounces is $2^a \times 3^b \times 5^c \times 7^d \times 11^e$ for integers a, b, c, d , and e . Find $|a| + |b| + |c| + |d| + |e|$.
4. There are n lily pads in a row labeled $1, 2, \dots, n$ from left to right. Fareniss the Frog picks a lily pad at random to start on, and every second she jumps to an adjacent lily pad; if there are two such lily pads, she is twice as likely to jump to the right as to the left. After some finite number of seconds, there exists two lily pads A and B such that Fareniss is more than 1000 times as likely to be on A as she is to be on B . What is the minimal number of lily pads n such that this situation must occur?
5. A Princeton slot machine has 100 pictures, each equally likely to occur. One is a picture of a tiger. Alice and Bob independently use the slot machine, and each repeatedly makes independent plays. Alice keeps playing until she sees a tiger, at which point she stops. Similarly, Bob keeps playing until he sees a tiger. Given that Bob played twice longer than Alice, let the expected number of plays for Alice be $\frac{a}{b}$ with a, b relatively prime positive integers. Find the remainder when $a + b$ is divided by 1000.
6. Alice, Bob, and Carol are playing a game. Each turn, one of them says one of the 3 players' names, chosen from $\{\text{Alice, Bob, Carol}\}$ uniformly at random. Alice goes first, Bob goes second, Carol goes third, and they repeat in that order. Let E be the expected number of names that have been said when, for the first time, all 3 names have been said twice. If $E = \frac{m}{n}$ for relatively prime positive integers m and n , find $m + n$. (Include the last name to be said twice in your count.)
7. Cassidy has string of n bits, where n is a positive integer, which initially are all 0s or 1s. Every second, Cassidy may choose to do one of two things:
 1. Change the first bit (so the first bit changes from a 0 to a 1, or vice versa)
 2. Change the first bit after the first 1.
 Let M be the minimum number of such moves it takes to get from $1 \dots 1$ to $0 \dots 0$ (both of length 12), and N the number of starting sequences with 12 bits that Cassidy can turn into all 0s. Find $M + N$.
8. Physicists at Princeton are trying to analyze atom entanglement using the following experiment. Originally there is one atom in the space and it starts splitting according to the following procedure. If after n minutes there are atoms a_1, \dots, a_N , in the following minute every atom a_i splits into four new atoms, $a_i^{(1)}, a_i^{(2)}, a_i^{(3)}, a_i^{(4)}$. Atoms $a_i^{(j)}$ and $a_k^{(j)}$ are entangled if and only

P U M . C



if the atoms a_i and a_k were entangled after n minutes. Moreover, atoms $a_i^{(j)}$ and $a_k^{(j+1)}$ are entangled for all $1 \leq i, k \leq N$ and $j = 1, 2, 3$. Therefore, after one minute there is 4 atoms, after two minutes there are 16 atoms and so on.

Physicists are now interested in the number of unordered quadruplets of atoms $\{b_1, b_2, b_3, b_4\}$ among which there is an odd number of entanglements. What is the number of such quadruplets after 3 minutes?

Remark. Note that atom entanglement is not transitive. In other words, if atoms a_i, a_j are entangled and if a_j, a_k are entangled, this does not necessarily mean that a_i and a_k are entangled.