P U M ∴ C

Combinatorics B

1. A nonempty word is called pronounceable if it alternates in vowels (A, E, I, O, U) and consonants (all other letters) and it has at least one vowel. How many pronounceable words can be formed using the letters P, U, M, A, C at most once each? Words of length shorter than 5 are allowed.

2. Neel and Roshan are going to the Newark Liberty International Airport to catch separate flights. Neel plans to arrive at some random time between 5:30 am and 6:30 am, while Roshan plans to arrive at some random time between 5:40 am and 6:40 am. The two want to meet, however briefly, before going through airport security. As such, they agree that each will wait for n minutes once he arrives at the airport before going through security. What is the smallest n they can select such that they meet with at least 50% probability? The answer will be of the form $a + b\sqrt{c}$ for integers a, b, and c, where c has no perfect square factor other than 1. Report $a + b + c$.

3. Select two distinct diagonals at random from a regular octagon. What is the probability that the two diagonals intersect at a point strictly within the octagon? Express your answer as $a + b$, where the probability is $\frac{a}{b}$ and a and b are relatively prime positive integers.

4. Eighteen people are standing in a (socially-distanced) line to enter a grocery store. Five people are wearing a black mask, 6 are wearing a gray mask, and 7 are wearing a white mask. Suppose that these 18 people got on line in a random order. The expected number of pairs of adjacent people wearing different-colored masks can be given by $\frac{a}{b}$, where $\gcd(a, b) = 1$. Compute $a + b$.

5. Nelson is having his friend drop his unique bouncy ball from a 12 foot building, and Nelson will only catch the ball at the peak of its trajectory between bounces. On any given bounce, there is an 80% chance that the next peak occurs at $\frac{1}{3}$ the height of the previous peak and a 20% chance that the next peak occurs at 3 times the height of the previous peak (where the first peak is at 12 feet). If Nelson can only reach 4 feet into the air and will catch the ball as soon as possible, the probability that Nelson catches the ball after exactly 13 bounces is $2^a \times 3^b \times 5^c \times 7^d \times 11^e$ for integers a, b, c, d, and e. Find $|a| + |b| + |c| + |d| + |e|$.

6. There are n lilypads in a row labeled $1, 2, ..., n$ from left to right. Fareniss the Frog picks a lilypad at random to start on, and every second she jumps to an adjacent lilypad; if there are two such lilypads, she is twice as likely to jump to the right as to the left. After some finite number of seconds, there exists two lilypads A and B such that Fareniss is more than 1000 times as likely to be on A as she is to be on B. What is the minimal number of lilypads n such that this situation must occur?

7. A Princeton slot machine has 100 pictures, each equally likely to occur. One is a picture of a tiger. Alice and Bob independently use the slot machine, and each repeatedly makes independent plays. Alice keeps playing until she sees a tiger, at which point she stops. Similarly, Bob keeps playing until he sees a tiger. Given that Bob plays twice as much as Alice, let the expected number of plays for Alice be $\frac{a}{b}$ with a, b relatively prime positive integers. Find the remainder when $a + b$ is divided by 1000.

8. Alice, Bob, and Carol are playing a game. Each turn, one of them says one of the 3 players' names, chosen from {Alice, Bob, Carol} uniformly at random. Alice goes first, Bob goes second, Carol goes third, and they repeat in that order. Let E be the expected number of names that are have been said when, for the first time, all 3 names have been said twice. If $E = \frac{m}{n}$ for relatively prime positive integers m and n, find $m + n$. (Include the last name to be said twice in your count.)