P U M ∴ C

Team Round

The Team Round consists of 15 questions. Your team has **40 minutes** to complete the Team Round. Each problem is worth 5 points. At the conclusion of the test, in addition to providing answers to the questions, you will be asked to give a closed interval $[a, b]$, where a, b are nonnegative integers, for your projected number of correctly solved problems on the team round. If the number of correct solutions your team submitted lies in the proposed interval, the number of bonus points you will receive is $\frac{5}{|b-a|+1}$. Good luck!

1. An evil witch is making a potion to poison the people of PUMAClandia. In order for the potion to work, the number of poison dart frogs cannot exceed 5, the number of wolves' teeth must be an even number, and the number of dragon scales has to be a multiple of 6. She can also put in any number of tiger nails. Given that the stew has exactly 2021 ingredients, in how many ways can she add ingredients for her potion to work?

2. Let $k \in \mathbb{Z}_{>0}$ be the smallest positive integer with the property that $k\frac{\gcd(x,y)\gcd(y,z)}{\operatorname{lcm}(x,y^2,z)}$ is a positive integer for all values $1 \le x \le y \le z \le 121$. If k' is the number of divisors of k, find the number of divisors of k'.

3. Let $f(N) = N\left(\frac{9}{10}\right)^N$, and let $\frac{m}{n}$ denote the maximum value of $f(N)$, as N ranges over the positive integers. If m and n are relatively prime positive integers, find the remainder when $m + n$ is divided by 1000.

4. Abby and Ben have a little brother Carl who wants candy. Abby has 7 different pieces of candy and Ben has 15 different pieces of candy. Abby and Ben then decide to give Carl some candy. As Ben wants to be a better sibling than Abby, so he decides to give two more pieces of candy to Carl than Abby does. Let N be the number of ways Abby and Ben can give Carl candy. Compute the number of positive divisors of N.

5. Given a real number t with $0 < t < 1$, define the real-valued function $f(t, \theta) = \sum_{n=-\infty}^{\infty} t^{|n|}\omega^n$, where $\omega = e^{i\theta} = \cos\theta + i\sin\theta$. For $\theta \in [0, 2\pi)$, the polar curve $r(\theta) = f(t, \theta)$ traces out an ellipse E_t with a horizontal major axis whose left focus is at the origin. Let $A(t)$ be the area of the ellipse E_t. Let $A(\frac{1}{2}) = \frac{a\pi}{b}$, where a, b are relatively prime positive integers. Find $100a + b$.

6. Jack plays a game in which he first rolls a fair six-sided die and gets some number n; then, he flips a coin until he flips n heads in a row and wins, or he flips n tails in a row in which case he rerolls the die and tries again. What is the expected number of times Jack must flip the coin before he wins the game?

7. The roots of the polynomial $f(x) = x^8 + x^7 - x^5 - x^4 - x^3 + x + 1$ are all roots of unity. We say that a real number $r \in [0, 1)$ is *nice* if $e^{2i\pi r} = \cos 2\pi r + i\sin 2\pi r$ is a root of the polynomial f and if $e^{2i\pi r}$ has positive imaginary part. Let S be the sum of the values of nice real numbers r. If $S = \frac{p}{q}$ for relatively prime positive integers p, q, find $p + q$.

8. The new PUMaC tournament hosts 2020 students, numbered by the following set of labels $1, 2, \ldots, 2020$. The students are initially divided up into 20 groups of 101, with each division into groups equally likely. In each of the groups, the contestant with the lowest label wins, and the winners advance to the second round. Out of these 20 students, we chose the champion uniformly at random. If the expected value of champion's number can be written as $\frac{a}{b}$, where a, b are relatively prime integers, determine $a + b$.

9. Let AX be a diameter of a circle Ω with radius 10, and suppose that C lies on Ω so that $AC = 16$. Let D be the other point on Ω so $CX = CD$. From here, define D' to be the reflection of D across the midpoint of AC, and X' to be the reflection of X across the midpoint

of CD. If the area of triangle $CD'X'$ can be written as $\frac{p}{q}$, where p, q are relatively prime, find $p+q$.

10. Determine the number of pairs (a, b), where $1 \leq a \leq b \leq 100$ are positive integers, so that $\frac{a^3+b^3}{a^2+b^2}$ is an integer.

11. ABC is a triangle where $AB = 10$, $BC = 14$, and $AC = 16$. Let DEF be the triangle with smallest area so that DE is parallel to AB, EF is parallel to BC, DF is parallel to AC, and the circumcircle of ABC is DEF's inscribed circle. Line DA meets the circumcircle of ABC again at a point X. Find AX^2.

12. Given an integer a_0, we define a sequence of real numbers $a_0, a_1, \ldots$ using the relation

$$a_i^2 = 1 + i a_{i-1}^2,$$

for $i \geq 1$. An index j is called *good* if a_j can be an integer for some a_0. Determine the sum of the indices j which lie in the interval $[0, 99]$ and which are not good.

13. Given a positive integer n with prime factorization $p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, we define $f(n)$ to be $\sum_{i=1}^{k} p_i e_i$. In other words, $f(n)$ is the sum of the prime divisors of n, counted with multiplicities. Let M be the largest odd integer such that $f(M) = 2023$, and m the smallest odd integer so that $f(m) = 2023$. Suppose that $\frac{M}{m}$ equals $p_1^{e_1} p_2^{e_2} \cdots p_l^{e_l}$, where the e_i are all nonzero integers and the p_i are primes. Find $|\sum_{i=1}^{l}(p_i + e_i)|$.

14. Heron is going to watch a show with n episodes which are released one each day. Heron wants to watch the first and last episodes on the days they first air, and he doesn't want to have two days in a row that he watches no episodes. He can watch as many episodes as he wants in a day. Denote by $f(n)$ the number of ways Heron can choose how many episodes he watches each day satisfying these constraints. Let N be the 2021st smallest value of n where $f(n) \equiv 2 \mod 3$. Find N.

15. Let $\triangle ABC$ be an acute triangle with angles $\angle BAC = 70°$, $\angle ABC = 60°$, let D, E be the feet of perpendiculars from B, C to AC, AB, respectively, and let H be the orthocenter of ABC. Let F be a point on the shorter arc AB of circumcircle of ABC satisfying $\angle FAB = 10°$ and let G be the foot of perpendicular from H to AF. If $I = BF \cap EG$ and $J = CF \cap DG$, compute the angle $\angle GIJ$.