PUMaC 2018 Power Round:
"Life is a game I lost..."

November 17, 2018

"The Game gives you a Purpose. The Real Game is, to Find a Purpose." — Vineet Raj Kapoor

Rules and Reminders

1. Your solutions may be turned in in one of two ways:
   - You may email them to us at **pumac2018power@gmail.com** by 8AM Eastern Standard Time on the morning of PUMaC, November 17, 2018 with the subject line "PUMaC 2018 Power Round."
   - You may hand them in to us when your team checks in on the morning of PUMaC. Please staple your solutions together, including the cover sheet.

   The cover sheet (the last page of this document) should be the first page of your submission. Each page should have on it the **team number** (not team name) and **problem number**. This number can be found by logging in to the coach portal and selecting the corresponding team. Solutions to problems may span multiple pages, but include them in continuing order of proof.

2. You are encouraged, but not required, to use LaTeX to write your solutions. If you submit your power round electronically, **you may not submit multiple times**. The first version of the power round solutions that we receive from your team will be graded. If submitting electronically, **you must submit a PDF**. No other file type will be graded.

3. Do not include identifying information aside from your team number in your solutions.

4. Please collate the solutions in order in your solution packet. Each problem should start on a new page, and solutions should be written on one side of the paper only (there is a point deduction for not following this formatting).

5. On any problem, you may use without proof any result that is stated earlier in the test, as well as any problem from earlier in the test, even if it is a problem that your team has not solved. These are the **only** results you may use. You may not cite parts of your proof of other problems: if you wish to use a lemma in multiple problems, please reproduce it in each one.

6. When a problem asks you to "find with proof," "show," "prove," "demonstrate," or "ascertain" a result, a formal proof is expected, in which you justify each step you take, either by using a method from earlier or by *proving* that *everything* you do is correct. When a problem instead uses the word "explain," an informal explanation suffices. When a problem asks you to "find" or "list" something, no justification is required.

7. All problems are numbered as "Problem x.y.z" where x is the section number and y is the subsection. Each problem's point distribution can be found in parentheses before the problem statement.

8. **You may NOT use any references, such as books or electronic resources, unless otherwise specified. You may NOT use computer programs, calculators, or any other computational aids.**

9. Teams whose members use English as a foreign language may use dictionaries for reference.

10. **Communication with humans outside your team of 8 students about the content of these problems is prohibited.**

11. There are two places where you may ask questions about the test. The first is Piazza. Please ask your coach for instructions to access our Piazza forum. On Piazza, you may ask any question **so long as it does not give away any part of your solution to any problem**. If you ask a question on Piazza, all other teams will be able to see it. If such a question reveals all or part of your solution to a power round question, your team's power round score will be penalized severely. For any questions you have that might reveal part of your solution, or if you are not sure if your question is appropriate for Piazza, please email us at pumac@math.princeton.edu. We will email coaches with important clarifications that are posted on Piazza.

Introduction and Advice

The topic of this power round is **Combinatorial Game Theory**. A combinatorial game is a special type of game that is not commonly discussed in a typical Game Theory setting. Despite this, combinatorial games show up quite often; examples of complex combinatorial games include chess, go, and even tic-tac-toe. There are lots of unsolved questions in combinatorial game theory, and games such as chess still do not have a (discovered) optimal strategy.

Section 1 introduces you to a seemingly separate topic: surreal numbers. Although this is just Section 1, a lot of the definitions are difficult to grasp at first because of their recursive or inductive nature; do not worry. We gave a fairly lengthy dedication to this section, so you will be quite comfortable with surreal numbers by the end of the section.

Section 2 is an introduction to combinatorial games. Despite a few differences, you will notice many similarities between combinatorial games and surreal numbers. This section is fairly definition heavy, and we spend some time introducing games such as Toads and Frogs and Hackenbush.

Section 3 begins with a useful combinatorial game known as Nim. Next, you will learn about the Sprague-Grundy Theorem, a very important theorem in combinatorial game theory that links many games to Nim.

Section 4 provides some challenge problems on several combinatorial games. These problems will require you to use the material of previous sections in addition to lots of your own creativity.

This is not intended to be a complete course in Combinatorial Game Theory; in any event, a contest is far from the best way to provide a complete undertaking. After the Power Round is over, we advise you to read about topics from the round that interested you. We can give you recommended books to read as well (see the solutions)!

Here is some further advice with regard to the Power Round:

- **Read the text of every problem!** Many important ideas are included in problems and may be referenced later on. In addition, some of the theorems you are asked to prove are useful or even necessary for later problems.

- **Make sure you understand the definitions**. As we stated above, a lot of the definitions are not easy to grasp (especially in the Surreal Numbers section); don't worry if it takes you a while to understand them. If you don't, then you will not be able to do the problems. Feel free to ask clarifying questions about the definitions on Piazza (or email us).

- **Don't make stuff up**: on problems that ask for proofs, you will receive more points if you demonstrate legitimate and correct intuition than if you fabricate something that *looks* rigorous just for the sake of having "rigor."

- **Check Piazza often!** Clarifications will be posted there, and if you have a question it is possible that it has already been asked and answered in a Piazza thread (and if not, you can ask it, assuming it does not reveal any part of your solution to a question). **If in doubt about whether a question is appropriate for Piazza, please email us at pumac@math.princeton.edu.**

Good luck, and have fun!

– Nathan Bergman & Jackson Blitz

We'd like to acknowledge and thank many individuals and organizations for their support; without their help, this Power Round (and the entire competition) could not exist. Please refer to the solutions of the power round for full acknowledgments.

Contents

Notation

- $\forall$: for all. *ex.:* $\forall x \in \{1, 2, 3\}$ *means "for all* x *in the set* $\{1, 2, 3\}$*"*
- $A \subset B$: proper subset. *ex.:* $\{1, 2\} \subset \{1, 2, 3\}$, *but* $\{1, 2\} \not\subset \{1, 2\}$
- $A \subseteq B$: subset, possibly improper. *ex.:* $\{1\}, \{1, 2\} \subseteq \{1, 2\}$
- $f : x \mapsto y$: f maps x to y. *ex.: if* $f(n) = n - 3$ *then* $f : 20 \mapsto 17$ *and* $f : n \mapsto n - 3$ *are both true.*
- $\{x \in S : C(x)\}$: the set of all x in the set S satisfying the condition $C(x)$. *ex.:* $\{n \in \mathbb{N} : \sqrt{n} \in \mathbb{N}\}$ *is the set of perfect squares.*
- $\mathbb{N}$: the natural numbers, $\{1, 2, 3, \dots\}$.
- $\mathbb{Z}$: the integers.
- $\mathbb{R}$: the real numbers.
- $\mathbb{D}$: the dyadic rationals, $\{\frac{m}{2^n} : m \in \mathbb{Z}, n \in \mathbb{N} \cup \{0\}\}$.

1 Surreal Numbers (93 points)

The surreal numbers provide a recursive way to construct a number system. They have many properties that will be useful in analyzing combinatorial games, so we investigate them below. Our goal is to model the surreal numbers to emulate properties of the real numbers.

1.1 Defining the Surreal Numbers (41 points)

We define the *surreal numbers* recursively in stages called *days*, together with a strict total ordering $<$ between them at each stage. The idea is that every surreal number x will take the form $x = \{L \mid R\}$, where L and R are any subset of surreal numbers appearing on previous days, with every element of L less than every element of R, with respect to the order relation defined below. We denote L_x and R_x to be the sets L and R of $x = \{L \mid R\}$, respectively. Note L_x and R_x depend on the form of $x = \{L \mid R\}$. We also denote x^L and x^R to be an arbitrary element of L_x and R_x respectively. We define an order relation using recursive definitions below.

Definition 1.1.A. For two surreal numbers x and y we let $x \geq y$ if and only if there does not exist an element $a \in R_x$ such that $a \leq y$ and there does not exist an element $b \in L_y$ such that $b \geq x$. Let $x \leq y$ if and only if $y \geq x$.

Definition 1.1.B. Let $x = y$ if and only if $x \geq y$ and $y \geq x$. Let $x > y$ if and only if $x \geq y$ and $y \not\geq x$. Let $x < y$ if and only if $y > x$.

Definition 1.1.C. (The critical condition) For all $a \in L_x$ and for all $b \in R_x$, $a < x < b$.

We make two important notes about the definitions above. First, it is important to note that some surreal numbers can have more than one form depending on which elements are in L and R (more on this later). Second, the definitions above are inductive, because to show something is true for x and y, we have to assume statements about x^L, x^R, y^L, and y^R. Note here x^L denotes any element in L_x, with analogous notation for the other three terms. In the same way, surreal numbers are "invented" inductively.

Definition 1.1.D. We say a number is *born on day n* if its earliest construction occurs on day n.

On day 0, we start with the single surreal number $\{|\}$, which we denote by 0. Then, on the n^{th} day, we introduce surreal numbers of the form $x = \{L \mid R\}$, where L and R are any subset of surreal numbers appearing on any previous day, and for all $a \in L_x$ and for all $b \in R_x$, we have $a < b$.

0 satisfies the conditions for surreal numbers written above. $0 = \{|\}$, so $L_0 = R_0 = \{\}$. Hence, every element in L_0 is less than every element in R_0, which means 0 is a surreal number.

Let us demonstrate some of our definitions. For example, $0 \geq 0$ because there is no $a \in R_0$ such that $a \leq 0$ and no $b \in L_0$ such that $b \geq 0$.

0 is the only number born on day 0. Now that we defined 0, for any new surreal number $x = \{L \mid R\}$ born on day 1, we have L and R can be either the empty set or

the set just containing 0. This enables us to create four potential new surreal numbers: $\{|\}, \{0\ |\}, \{|\ 0\}, \{0\ |\ 0\}$. Note that $\{|\}$ is not a new surreal number, because we already defined $0 = \{|\}$. Define $*$ to be $\{0 \mid 0\}$. $*$ contradicts the definition of the surreal numbers, as $0 \in L_*$ and $0 \in R_*$, yet $0 \not< 0$. Hence, $*$ is not a surreal number. So on day 1, we have two new surreal numbers, which we denote $1 = \{0\ |\}$ and $-1 = \{|\ 0\}$. You will prove that 1 and -1 satisfy the properties of surreal numbers later. We will now show that these three surreal numbers follow the desired order of the real numbers: $-1 < 0 < 1$. First, note that $0 \not\geq 1$, because there exists $a \in L_1$ such that $a \geq 0$, namely $a = 0$. By a similar method, we can show that $1 \geq 0$, and putting these two together we get that $1 > 0$. You will prove that $-1 < 0$ and $-1 < 1$ below.

On day 2, we now have a total of 8 sets to use for L and R: $\{\}$, $\{-1\}$, $\{0\}$, $\{1\}$, $\{-1, 0\}$, $\{0, 1\}$, $\{-1, 1\}$, and $\{-1, 0, 1\}$. However, not all combinations of these will yield new surreal numbers. We cannot have an element in $a \in R_x$ that is less than or equal to an element in $b \in L_x$, and we have a few repeat surreal numbers that were born on days 0 and 1. We also end up with a few equivalent forms for certain surreal numbers (more on this later). For example, on day 2, the following forms are all representative of the surreal number 0:

$$0 = \{|\} = \{-1\ |\} = \{|\ 1\} = \{-1 \mid 1\}$$

The following forms are all representative of the surreal number we denote as -2 born on day 2:

$$-2 = \{|\ -1\} = \{|\ -1, 0\} = \{|\ -1, 1\} = \{|\ -1, 0, 1\}$$

There are three other surreal numbers born on day 2, which we denote as follows:

$$2 = \{1\ |\} = \{0, 1\ |\} = \{-1, 1\ |\} = \{-1, 0, 1\ |\}$$

$$\frac{1}{2} = \{0 \mid 1\} = \{-1, 0 \mid 1\}$$

$$-\frac{1}{2} = \{-1 \mid 0\} = \{-1 \mid 0, 1\}$$

These surreal numbers satisfy their respective inequalities in the real numbers. The assignments of certain surreal numbers like 2 and $\frac{1}{2}$ above may seem arbitrary at the moment. However, these assignments will make sense in conjunction with the definitions of operations on surreal numbers such as addition and multiplication defined later.

In general, surreal numbers born on day n are of the form $\{L \mid R\}$, where L and R are any subset of surreal numbers formed on day $n - 1$ satisfying the critical condition (which includes both those surreal numbers born on day $n - 1$ and all surreal numbers born before day $n - 1$).

Problem 1.1.1. (5 points)

a) Prove for all surreal numbers x that $a \in L_x$ implies $a \notin R_x$.

b) Prove that 1 is a surreal number.

c) Prove that -1 is a surreal number.

d) Demonstrate $-1 < 0$.

e) Demonstrate $-1 < 1$.

Now we shall define binary operations on the surreal numbers.

Definition 1.1.E. Let a surreal number x be *positive* if $x > 0$. Let a surreal number y be *negative* if $y < 0$.

Definition 1.1.F. Let $-x = \{-x^R \mid -x^L\}$.

Note that x^L and x^R iterate through all of L_x and R_x. For example, $2 = \{0, 1 \mid\} = \{1 \mid\}$. Because $0 = -0$ (you will show this below), $-2 = \{\mid -1, 0\} = \{\mid -1\}$, which is consistent with our definition of -2 above.

Definition 1.1.G. Define addition on surreal numbers as follows

$$x + y = \{x^L + y, x + y^L \mid x^R + y, x + y^R\}$$

Definition 1.1.H. Define subtraction on surreal numbers as $x - y = x + (-y)$.

Definition 1.1.I. Define multiplication on surreal numbers as follows

$$x \cdot y = \{x^L y + x y^L - x^L y^L, x^R y + x y^R - x^R y^R \mid x^L y + x y^R - x^L y^R, x^R y + x y^L - x^R y^L\}$$

We include these definitions of operations here so that you can confirm that the placement of surreal numbers such as 2 and $\frac{1}{2}$ as they are defined above indeed preserve the arithmetic operations of the real numbers. You may also assume that multiplying or adding a number by the empty set gives the empty set. Confirm a few properties of the surreal numbers below.

Problem 1.1.2. (5 points)

a) Demonstrate $0 = -0$.

b) Demonstrate $-1 = -(1)$, proving notation is consistent for -1.

c) Demonstrate $0 + 0 = 0$.

d) Demonstrate $0 \cdot 0 = 0$.

e) Demonstrate $1 + 1 = 2$.

To recap, 0 is born on day zero. -1 and 1 are born on day one. -2, 2, $-\frac{1}{2}$, and $\frac{1}{2}$ are born on day two. We now generalize the surreal numbers born on day n. On day n, the greatest possible surreal number x would have all the previously born surreal numbers in L_x, so on day n, the greatest possible surreal number is x, defined as $x = n = \{0, 1, 2, 3, \ldots, n-1 \mid\}$. Similarly, the least surreal number born on day n is $-x = -n = \{\mid 0, -1, -2, -3, \ldots, -(n-1)\}$, and the smallest positive surreal number on day n is $\frac{1}{2^n} = \{0 \mid 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^{n-1}}\}$.

Definition 1.1.J. Let ω denote the first day after all finite days, and also let ω denote the surreal number $\omega = \{1, 2, 3, \cdots \mid\}$.

As it turns out, many surreal numbers are born on day ω. For example, it can be shown that $\frac{1}{3} = \{\frac{1}{4}, \frac{1}{4} + \frac{1}{16}, \frac{1}{4} + \frac{1}{16} + \frac{1}{64}, \ldots \mid \frac{1}{2}, \frac{1}{2} - \frac{1}{8}, \frac{1}{2} - \frac{1}{8} - \frac{1}{32}, \ldots\}$, and that it is born on day ω. We also denote ϵ as $\epsilon = \frac{1}{\omega}$.

The surreal numbers turn out to be actually well-defined. Further, every real number $x \in \mathbb{R}$ has a representation $x = \{L \mid R\}$ as a surreal number, including all rational, algebraic, and transcendental numbers. In fact, many more numbers than just real numbers can be surreal numbers, such as $\omega, -\omega^2, \omega^\omega$ and many more.

It can often be confusing to determine the exact value of a surreal number x based solely on the sets of L_x and R_x. If neither set contains any use of ω, the following set of conditions can help determine x given L_x and R_x.

1. If L_x and R_x are both empty, then $x = 0$.

2. If R_x is empty and there is some smallest integer $n \geq 0$ greater than every element of L_x, x is this integer n.

3. If R_x is empty and there is no integer n greater than every element of L_x, then $x = \omega$.

4. If L_x is empty and there is some greatest integer $n \leq 0$ less than every element of R_x, x is this integer n.

5. If L_x is empty and there is no integer n less than every element of R_x, then $x = -\omega$.

6. If L_x and R_x are both non-empty and there exists some dyadic rational greater than every element of L_x and less than every element of R_x, x is the oldest (i.e. born earliest) such dyadic rational.

7. If L_x and R_x are both non-empty and there does not exist some dyadic rational greater than every element of L_x and less than every element of R_x, but there exists some dyadic fraction y in L_x that is greater than or equal to every element of L_x, $x = y + \epsilon$.

8. If L_x and R_x are both non-empty and there does not exist some dyadic rational greater than every element of L_x and less than every element of R_x, but there exists some dyadic fraction y in R_x that is less than or equal to every element of R_x, $x = y - \epsilon$.

9. If L_x and R_x are both non-empty and every dyadic rational is greater than some element of R_x or less than some element of L_x, then x cannot be represented as a dyadic fraction. In this case, L_x and R_x must both converge to the same number. To calculate x, find what L_x and R_x converge to (as in the example with $\frac{1}{3}$ above).

For example, let $x = \{-3, -2, 1 \mid\}$. Then R_x is empty but L_x is not, and there is a smallest integer greater than every element of L_x (this integer is 2), so $x = 2$. This is condition 2 from the 9 conditions above.

For another example, let $x = \{\frac{1}{2}, 2 \mid 5\}$ Then both L_x and R_x are non-empty, and there is an oldest dyadic rational greater than every element of L_x and less than every element of R_x; in particular, 3 is the oldest such dyadic rational, so $x = 3$. This is condition 6 from the 9 conditions above.

If one of the two sets contains a use of ω, you can view ω as a constant and treat the problem the same as the cases without ω above. For example, $\{0, \omega \mid\}$ can be viewed as ω times the surreal number $\{0, 1 \mid\} = 2$, so $\{0, \omega \mid\} = 2 \cdot \omega = 2\omega$.

Using the conditions of how to denote a surreal number by its representation, each representation is assigned to a unique surreal number. This surreal number may have infinite different representations, but it shares no representation with a distinct surreal number.

In particular, every question asking to "find" a value actually has a unique answer. It should be noted that if a surreal number has a reduced form in its real number interpretation, it should be written in it reduced form. For example, it can be shown $\frac{1}{2} \leq \frac{2}{4}$ and $\frac{2}{4} \leq \frac{1}{2}$, so we have $\frac{1}{2} = \frac{2}{4}$.

Problem 1.1.3. (5 points)

a) Find the value of x if $x = \{2, 6 \mid\}$.

b) Find the value of x if $x = \{-10, -4 \mid 3, 8\}$.

c) Find the value of x if $x = \{-1, \frac{1}{2} \mid 2\}$.

d) Find the value of x if $x = \{-1, \frac{1}{2} \mid 1, 2\}$.

e) Find the value of x if $x = \{-\frac{5}{8}, -\frac{5}{16} \mid -\frac{1}{4}, \frac{7}{2}, \frac{729}{64}\}$.

Problem 1.1.4. (6 points) Prove that a surreal number is born on a finite day if and only if it is a dyadic rational.

Problem 1.1.5. (2 points) Prove that if a surreal number x is born on day n, where $n > 0$ and n is finite, then L_x or R_x contains a surreal number born on day $n - 1$.

Problem 1.1.6. (4 points)

a) Find the day the surreal number 2018 is born on.

b) Find the day the surreal number $-\frac{7}{2}$ is born on.

c) Find the day the surreal number $\frac{21}{8}$ is born on.

d) Find the day the surreal number $\frac{3}{5}$ is born on.

Problem 1.1.7. (4 points) Ascertain the number of distinct surreal numbers born on or before day n.

Problem 1.1.8. (6 points) Prove that π and e are both born on day ω.

Problem 1.1.9. (4 points)

a) Ascertain the value of $\{0 \mid \frac{1}{\omega}\}$.

b) Ascertain the value of $\{0 \mid \frac{1}{\omega}, \frac{1}{2\omega}, \frac{1}{4\omega}, \ldots\}$.

1.2 General Statements about Surreal Numbers (52 points)

As was stated earlier, the definitions and construction of the surreal numbers rely on previous surreal numbers, so the proofs largely rely on induction. The form of inductive proofs we discuss all involve reducing the problem to an empty set of conditions, so it is not necessary to address a base case. As an example proof, consider the following proof of the transitive property:

Theorem 1.2.I. If $x \geq y$ and $y \geq z$, then $x \geq z$.

Proof 1.2.1. $x \geq y$ implies there does not exist $a \in R_x$ such that $a \leq y$. We assume the theorem holds for x^R (for all in R_x), y, and z, so by induction we cannot have $a \in R_x$ such that $a \leq z$. By a similar argument, we cannot have $b \in L_z$ such that $x \leq b$, so $x \geq z$.

There are lots of properties of the surreal numbers that we want to prove. They were constructed to behave like the real numbers, so we want to prove many of the properties the real numbers have. We will now prove that the surreal numbers form an abelian group.

Definition 1.2.A. A *group* G is a set equipped with an operation $\cdot$ and a fixed "identity" element e such that for all $a, b, c \in G$

- $a \cdot b \in G$.
- $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- $a \cdot e = e \cdot a = a$.
- There exists an element $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$.

Definition 1.2.B. A group G is *abelian* if for all $a, b \in G$ we have $a \cdot b = b \cdot a$.

Theorem 1.2.II. Commutative Property of Addition: $x + y = y + x$.

Proof 1.2.2. $x + y = \{x^L + y, x + y^L \mid x^R + y, x + y^R\}$. By induction, we know that the pairs (x^L, y), (x, y^L), (x^R, y), (x, y^R) all satisfy commutativity, so

$$\{x^L + y, x + y^L \mid x^R + y, x + y^R\} = \{y + x^L, y^L + x \mid y + x^R, y^R + x\} = y + x$$

Problem 1.2.1. (4 points)

a) Prove $-(-x) = x$.

b) Prove that $-(x + y) = -x + (-y)$.

Problem 1.2.2. (8 points)

a) (Additive Identity) Prove that $x + 0 = 0 + x = x$.

b) (Associative Law of Addition) Prove that $(x + y) + z = x + (y + z)$.

c) (Additive Inverse) Prove that $x + (-x) = 0$.

Problem 1.2.3. (13 points)

a) Show that $x - x^L > 0$ and $x^R - x > 0$.

b) Prove that if $x > 0$ and $y > 0$ then $x \cdot y > 0$.

c) Show that our definition of multiplication is consistent by showing that $(x \cdot y)^L < (x \cdot y) < (x \cdot y)^R$.

d) Prove that $x \leq y$ if and only if $x + z \leq y + z$.

With these properties, we conclude that the surreal numbers with the operation addition form an abelian group. Next, we will prove some nice properties about multiplication of the surreal numbers.

Theorem 1.2.III. (Zero Multiplication) For all x, $x \cdot 0 = 0$.

Proof 1.2.3. Every 0, 0^L, and 0^R term are 0 or the empty set, so we get that $x \cdot 0 = \{|\} = 0$.

Theorem 1.2.IV. (Distributive Property) $(x + y)z = xz + yz$

Proof 1.2.4.

$$
\begin{gathered}
(x+y)z = \{(x+y)^L z + (x+y)z^L - (x+y)^L z^L, \dots \mid \dots\} = \\
= \{(x^L + y)z + (x+y)z^L - (x^L + y)z^L, \\
(x + y^L)z + (x+y)z^L - (x + y^L)z^L, \dots \mid \dots\} = \\
= \{(x^L z + x z^L - x^L z^L) + yz, xz + (y^L z + y z^L - y^L z^L), \dots \mid \dots\} \\
= xz + yz
\end{gathered}
$$

Problem 1.2.4. (15 points)

a) (Multiplicative Identity) Prove that $x \cdot 1 = x$.

b) (Commutative Property) Prove that $x \cdot y = y \cdot x$.

c) (Negative Multiplication) Prove that $(-x)y = x(-y) = -(x \cdot y)$.

d) (Associative Property) Prove that $(x \cdot y)z = x(y \cdot z)$.

e) (Zero Product Property) Prove that $x \cdot y = 0$ if and only if $x = 0$ or $y = 0$.

With these properties completed, we can conclude that the surreal numbers form a ring. Lastly, we will show they form a field by finding the inverse for a number x. If you do not know what a ring or field is, no need to worry. We will not ask about it.

We will now define one more useful term for operating on surreal numbers.

Definition 1.2.C. Let x be a positive surreal number. Define the multiplicative inverse of x, alternatively denoted by x^{-1}, to be

$$y = \left\{ 0, \frac{1 + (x^R - x)y^L}{x^R}, \frac{1 + (x^L - x)y^R}{x^L} \,\middle|\, \frac{1 + (x^L - x)y^L}{x^L}, \frac{1 + (x^R - x)y^R}{x^R} \right\}$$

In the definition above, we use the notation $\frac{a}{b}$ to denote $a(b^{-1})$. Note that the definition of y (like the definitions earlier) is inductive, so to find the inverse of x, it is necessary to know the inverses of x^L, x^R. You will now show that y is the inverse of x.

Problem 1.2.5. (12 points) In this problem, let $y = x^{-1}$.

a) Show for all $y^L \in L_y$ and $y^R \in R_y$ that $x \cdot y^L < 1 < x \cdot y^R$.

b) Show that y is a surreal number.

c) Show for all $(x \cdot y)^L \in L_{xy}$ and $(x \cdot y)^R \in R_{xy}$ that $(x \cdot y)^L < 1 < (x \cdot y)^R$.

d) Show that $x \cdot y = 1$.

Hence, the surreal numbers create an inductive representation of the real numbers. This notion will be very useful in the next section while examining combinatorial games.

2 Introduction to Combinatorial Game Theory (63 points)

Combinatorial game theory is the study of all turn-based games with perfect information, meaning all players know every possible move by the opponent and every previous event at any time. To best understand the terminology in this section, it will be beneficial to look for similarities between combinatorial games and surreal numbers.

2.1 Combinatorial Game Definitions (3 points)

Definition 2.1.A. A (two-player) *combinatorial game* consists of a space of possible positions, together with a specification of which positions each player can move to on their own turn. The game ends if a player has no legal moves at some position.

Definition 2.1.B. A *game* is an individual position in a combinatorial game.

Confusingly, the individual positions are also called games, so a game (i.e. position) can be written as $G = \{L_G \mid R_G\}$, where L_G is the set of games that the left player can move to, and R_G is the set of games that the right player can move to. See Definition 2.1.J for clarity on this notation and the left and right player.

Definition 2.1.C. *Normal play* is when the player who makes the last move wins. *Misére play* is when the player who makes the last move loses.

Unless otherwise stated, normal play will be assumed.

Definition 2.1.D. Combinatorial games where both players have identical move sets are *impartial*.

Definition 2.1.E. *N-positions* are when the first player can guarantee a win. *P-positions* are when the second player can guarantee a win.

N-positions and P-positions get their name from being good for the next player and previous player, respectively.

Definition 2.1.F. A *subgame* of a game G is a game which can occur after some set of moves are performed from G.

Definition 2.1.G. A game G is *finite* if it only has finite subgames.

Definition 2.1.H. A game G is *loopfree* if there does not exist a sequence of moves from G that repeats a game.

Definition 2.1.I. A game G is *short* if it is finite and loopfree.

We note combinatorial games have a left player and a right player, each with a possibly distinct set of moves, which coincide with the first and second player, not necessarily respectively.

Definition 2.1.J. We may write the *Left and Right Options* of a game G as L_G and R_G, denoting the list of possibilities to move for the left and right player, respectively. We may write $G = \{L_G \mid R_G\}$. We also denote G^L and G^R to be an arbitrary element of L_G and R_G respectively.

Note the left player and right player are arbitrarily assigned to the players. Typically, the left player is the first player and the right player is the second player, but this is not necessarily the case.

Definition 2.1.K. *L-positions* are when the left player can guarantee a win, no matter who moves first. *R-positions* are when the right player can guarantee a win, no matter who moves first.

Definition 2.1.L. $G > 0$ (G is *positive*) if there is a winning strategy for the Left player.

Definition 2.1.M. $G < 0$ (G is *negative*) if there is a winning strategy for the Left player.

Definition 2.1.N. $G = 0$ (G is *zero*) if there is a winning strategy for the second player.

Definition 2.1.O. $G \,||\, 0$ (G is *fuzzy*) if there is a winning strategy for the first player.

We also have $G \geq 0$ if $G = 0$ or $G > 0$ and $G \leq 0$ if $G = 0$ or $G < 0$. Now, we define a particularly special game.

Definition 2.1.P. Denote by 0 the empty game with no options, $0 = \{|\}$.

For an example of this notation, we define our first combinatorial game.

Game Definition 2.1.I. *Toads and Frogs* is played on a $1 \times n$ strip of squares. At all times, each square is either empty or occupied by a single toad or frog. The left player may move a toad one square to the right if it is empty. If a frog occupies the space immediately to a toad's right, and the space immediately right of the frog is empty, the left player may move the toad into that empty space. This move is called a "hop." Toads may not hop over more than one frog or another toad. Similarly, the right player may move frogs left in the same fashion. The first player to be unable to move loses (normal play).

If we have a game of Toads and Frogs G without any moves played so far on a 1×6 strip with a toad in the 1st square and frogs in the 4th and 6th squares, we will denote this game as T □□ F □ F. We will use this notation throughout the rest of the power round for Toads and Frogs, with X^n denoting any position X repeated n times side by side. We may write $G = \{T_2 \mid F_3, F_5\}$ where T_2 =□ T □ F □ F, F_3 =T □ F □□ F, and F_5 =T □□ F F □. Note that G does not know whose move it is, so even if G is only possible on one player's move (for example, the start of the game) we still write both.

Problem 2.1.1. (3 points) Prove which player can guarantee a win in Toad and Frogs played on a 1×6 strip with a toad in the 1st square and frogs in the 4th and 6th squares.

2.2 $\tilde{\mathbb{G}}$ (22 points)

We can formally build a group $\tilde{\mathbb{G}}$ of short games with rich structure. We can define some useful sets formally.

Definition 2.2.A. Define $\tilde{\mathbb{G}}_0 = \{0\}$. For $n \geq 0$, define

$$\tilde{\mathbb{G}}_{n+1} = \{\{L_G \mid R_G\} : L_G, R_G \subseteq \tilde{\mathbb{G}}_n\}$$

Definition 2.2.B. Define $\tilde{\mathbb{G}} = \bigcup_{n \geq 0} \tilde{\mathbb{G}}_n$. A game G is *short* if $G \in \tilde{\mathbb{G}}$.

Definition 2.2.B formalizes the definition of a short game.

Problem 2.2.1. (10 points) Prove Definition 2.1.I and Definition 2.2.B are both equivalent definitions of short games.

Recall that $*$ was defined to be $\{0 \mid 0\}$ in Section 1. Although $*$ is not a surreal number, it does in fact represent a combinatorial game: the game where both players only have the option of moving to the 0 game. $*$ is an unconditional first-player win, so it is an example of a fuzzy game. $*$ is just one example of a combinatorial game that cannot be expressed as a surreal number.

Problem 2.2.2. (2 points)

a) Explain why the 0 game is a 2^{nd} player win.

b) Explain why the $*$ game is a 1^{st} player win.

Problem 2.2.3. (4 points) (Fundamental Theorem) Let G be short and assume normal play. Prove that either the left player can force a win playing first or else the right player can force a win playing second, but not both.

Before we continue, we should go on a slight digression explaining why the surreal numbers were introduced before any of the theme of the power round. The surreal numbers have a very close connection to combinatorial game theory. Games are represented in the same form $\{L \mid R\}$ as surreal numbers. Every surreal number denoted by a real number is a game. However, not every game is a surreal number, as games need not satisfy the critical condition.

Nevertheless, the theory of surreal numbers prove to be helpful in examining games. We shall define addition and negation of short games exactly as surreal numbers. All short games turn out to be represented by a dyadic rational, which makes sense as short games are finite and dyadic rationals are born on a finite day.

Hence, in proofs from now on, where appropriate, one may reference the theory of surreal numbers.

We now introduce Hackenbush as an application of the concepts above.

Game Definition 2.2.I. The game of *Hackenbush* starts with a line on the ground and a finite series of red and blue line segments (some connected to each other) all connected either directly or indirectly to the ground line. Two players take turns removing segments of their corresponding color (the Left player moves first and can only remove blue segments, the Right player can only remove red segments). At the end of a turn, any segments no longer connected to the ground are also removed. The first person who cannot delete a segment on their turn loses. Note that some variants of Hackenbush include green line segments both players can remove, and in some variants all line segments are removable by both players, but we do not consider those forms here.

The simplest Hackenbush game is just the line on the ground. In this variant, neither player has any moves to make, so $L_G = R_G = \{\}$. As was the case with the surreal numbers, the empty game is the 0 game. A couple other examples of Hackenbush games are displayed below.

Problem 2.2.4. (4 points)

a) Is Hackenbush a game of normal play or misére play?

b) Is Hackenbush an impartial game?

c) Prove that Hackenbush is a short game.

Definition 2.2.C. If G and H are short games, then we define the *disjunctive sum* as

$$G + H = \{G^L + H, G + H^L \mid G^R + H, G + H^R\}$$

The disjunctive sum is clearly commutative and associative. Note that the disjunctive sum of short games is a short game.

Definition 2.2.D. Let G be short. Then define the negation of G by $-G = \{-G^R | -G^L\}$.

For example, in the game of Hackenbush, the game $-G$ is obtained by turning all red lines blue and all blue lines red. Note that G is a short game if and only if $-G$ is a short game.

Problem 2.2.5. (2 points)

a) Prove for any short game G we have $-(-G) = G$.

b) Evaluate $* + *$.

2.3 $\mathbb{G}$ (38 points)

Definition 2.3.A. Denote by $o(G)$ the outcome of the game G, which by the fundamental theorem exists as one of four options for short games: a first player win, a second player win, a left player win, or a right player win (meaning G is a N-position, P-position, L-position, or R-position, respectively).

Definition 2.3.B. For $G, H \in \tilde{\mathbb{G}}$, we say $G = H$ if $o(G + X) = o(H + X)$ for all $X \in \tilde{\mathbb{G}}$.

Definition 2.3.C. An *equivalence relation* $\sim$ satisfies for all a, b, c

- $a \sim a$.
- $a \sim b$ if and only if $b \sim a$.
- If $a \sim b$ and $b \sim c$ then $a \sim c$.

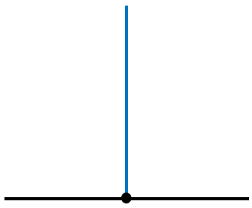
Problem 2.3.1. (3 points) Prove that $=$ (in the context of games) is an equivalence relation.

Definition 2.3.D. The *game value* of G is its equivalence class modulo $=$. The set of game values is $\mathbb{G}$. For notational convenience, we will sometimes use G to denote both a game and its corresponding game value when this introduces no ambiguity.

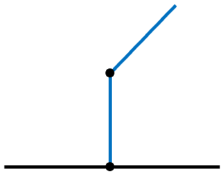
Problem 2.3.2. (6 points) Let G be an impartial game. Prove G is equivalent to 0 if and only if G is a P-position.

Problem 2.3.3. (5 points) Prove $\mathbb{G}$ is an abelian group under addition.

We will now delve deeper into the notion of game value by using the combinatorial game of Hackenbush. As the name suggests, the 0 game (discussed earlier) has value $G = 0$. To construct more complicated games of Hackenbush, we use a similar approach to our construction of the surreal numbers.

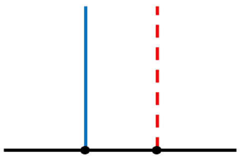


Hackenbush Figure 1



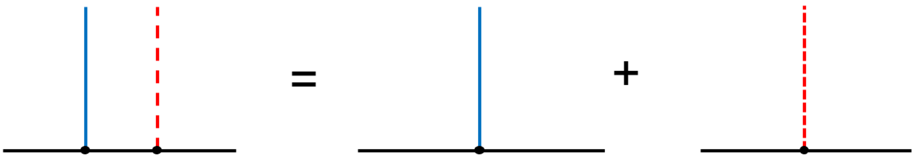
Hackenbush Figure 2

In Hackenbush Figure 1, Left player can delete the blue edge, creating the 0 game, while Right player has no moves. So $L_G = \{0\}, R_G = \{\}$ and the game in Figure 1 corresponds to value $G = \{0\ |\} = 1$, because the surreal number $\{0\ |\} = 1$. In Hackenbush Figure 2, Left player can delete the top blue edge, creating the game 1, or the bottom blue edge, creating the 0 game. Right player still has no moves, so this game has value $G = \{0, 1\ |\} = 2$.

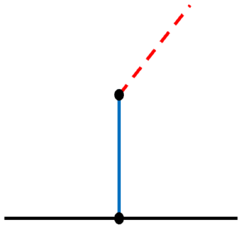


Hackenbush Figure 3

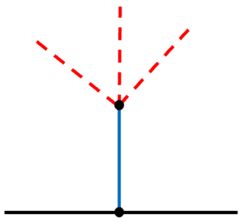
In Hackenbush Figure 3, Left player can delete the blue edge, creating the game -1, and Right player can delete the red edge, creating the game 1. So this game has value $G = \{-1 \mid 1\} = 0$, which is another game of value 0. Note that this game is essentially two different games: the game of one red line (which has value -1) and the game of one blue line (which has value 1); it is no coincidence that $1 + (-1) = 0$, the value of the combined game. Generally speaking, if G_1 and G_2 are the values of two games, then the combined game has value $G_1 + G_2$.



In general, the game value can say a lot about who is likely to win the game. If $G > 0$, then there is a winning strategy for the Left player. If $G < 0$, then there is a winning strategy for the Right player. If $G = 0$, then depending on the game either player could win. Games can have fractional values as well. For example, consider the games below.



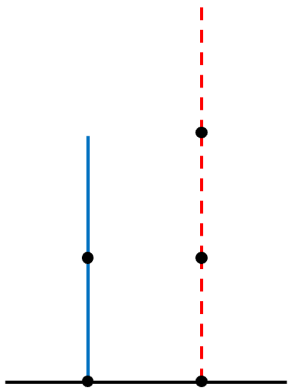
Hackenbush Figure 4



Hackenbush Figure 5

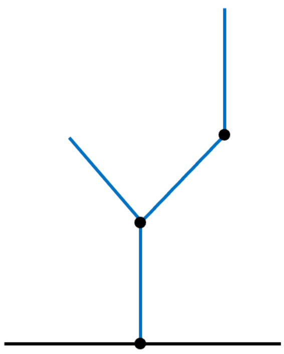
In Hackenbush Figure 4, Left player's only move is to delete the blue edge, creating the game 0, and Right player can delete the red edge, creating the game -1. So $G = \{0|1\} = \frac{1}{2}$. For our last example, we look at a game where the values must be found recursively, in the same way that surreal numbers born on day n are formed from numbers born earlier. In Hackenbush Figure 5, left player's only move is to create game 0. Right player has three possible lines to remove, but each leaves the same game G': two red lines connected to the blue line. The value of G' is not immediately obvious: for this game, left player can create game 0, while right player can create the game with one red line on top of one blue line (which is Hackenbush Figure 4 and has value $\frac{1}{2}$. So $G' = \{0 \mid \frac{1}{2}\} = \frac{1}{4}$, and $G = \{0 \mid \frac{1}{4}\} = \frac{1}{8}$.

Problem 2.3.4. (2 points) Demonstrate the game value of Hackenbush Game 1 below.



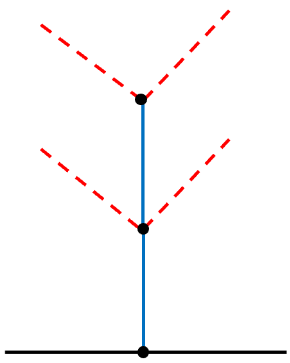
Hackenbush Game 1

Problem 2.3.5. (2 points) Demonstrate the game value of Hackenbush Game 2 below.



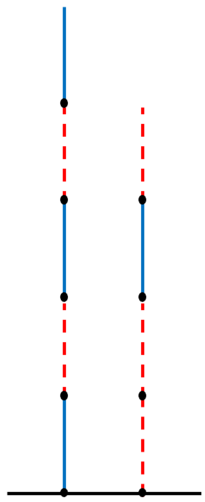
Hackenbush Game 2

Problem 2.3.6. (4 points) Demonstrate the game value of Hackenbush Game 3 below.



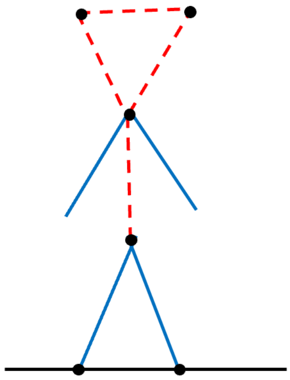
Hackenbush Game 3

Problem 2.3.7. (3 points) Demonstrate the game value of Hackenbush Game 4 below.



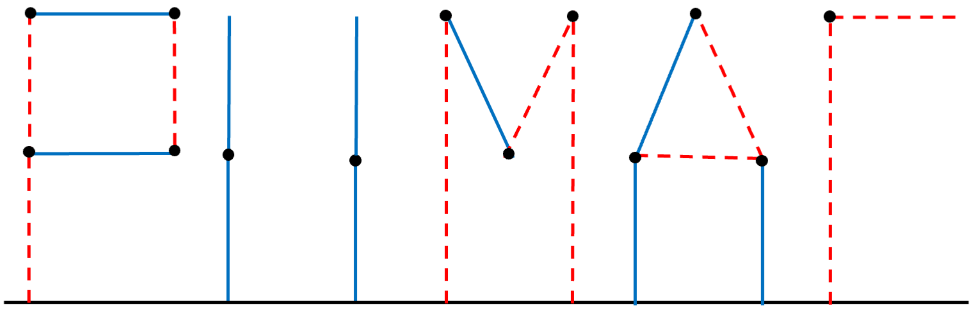
Hackenbush Game 4

Problem 2.3.8. (5 points) Demonstrate the game value of Hackenbush Game 5 below.



Hackenbush Game 5

Problem 2.3.9. (8 points) Demonstrate the game value of Hackenbush Game 6 below.



Hackenbush Game 6

3 Nim and the Sprague-Grundy Theorem (92 points)

3.1 Nim (18 points)

Nim is quite possibly the canonical combinatorial game in the topic of combinatorial game theory.

Game Definition 3.1.I. In *Nim* two players are presented with an arbitrary number of piles of arbitrary numbers of tokens. On their turn, a player may take as many tokens as they wish from any one pile. Whoever removes the last token wins.

Note that Nim is the disjunctive sum of its heaps. Now we present some definitions about the classic game.

Definition 3.1.A. The *nim-sum* of two nonnegative integers a and b is denoted by $a \oplus b$ and is obtained by "adding without carrying"/"exclusive or" in binary.

For example, $5 \oplus 7 = 101_2 \oplus 111_2 = 010_2 = 2$.

Definition 3.1.B. The *nim-value* of a game of Nim G is $a_1 \oplus a_2 \oplus \ldots \oplus a_n$ where $a_1, a_2, \ldots, a_n$ are the size of the token piles in G.

Problem 3.1.1. (4 points)

a) Find the nim-value of a game of Nim with piles of token size $5, 6, 2, 9$, and 2018.

b) Find the nim-value of a game of Nim with 2018 piles of token size 2018.

Definition 3.1.C. G is a *zero position* if its nim-value is 0.

These definitions allow for an easy formulation of the general winning strategy of the game of Nim, with the help of a powerful theorem.

Problem 3.1.2. (6 points) (Bouton's Theorem) Let G be a Nim position. Prove that if G is a zero position, then every move from G leads to a nonzero position. Prove that if G is a nonzero position, then there exists a move from G to a zero position.

Problem 3.1.3. (4 points) Find with proof the N-positions and P-positions of Nim.

Definition 3.1.D. A *nimber* is the game of Nim of a single heap of size n, and is denoted by $*n$. Note $0 = *0$ and let $*$ denote $*1$.

Note $*n = \{0, *, *2, \ldots, *(n-1) \mid 0, *, *2, \ldots, *(n-1)\}$.

Problem 3.1.4. (4 points) Prove for all $a, b \in \mathbb{N}$ we have $*a + *b = *(a \oplus b)$.

3.2 Nim Variants (41 points)

Problem 3.2.1. (5 points) In a game of misére nim, find with proof the N-positions and P-positions (recall the definition of misére from a previous section).

Game Definition 3.2.I. *Triple Nim* is played with the same rules as Nim, except a player may take from up to three piles (and at least one pile), rather than just one. The player may remove an arbitrary number from each pile.

Problem 3.2.2. (8 points) Find with proof the N-positions and P-positions of normal play Triple Nim.

Problem 3.2.3. (4 points) Find with proof the N-positions and P-positions of misére play Triple Nim.

Game Definition 3.2.II. In (n,r)-*Nim* there is only one pile of n coins. Each player can take up to r coins from the pile.

Problem 3.2.4. (5 points) Find with proof the N-positions and P-positions of normal play (n,r)-Nim.

Problem 3.2.5. (5 points) Find with proof the N-positions and P-positions of misére play (n,r)-Nim.

Game Definition 3.2.III. In *Tiger Nim* there is only one pile of coins. The first player may take up to all but one of the coins on the first turn. Every subsequent move can remove up to twice the number of coins taken on the previous move.

Problem 3.2.6. (12 points) Find with proof the N-positions and P-positions of normal play Tiger Nim.

Problem 3.2.7. (2 points) Find with proof the N-positions and P-positions of misére play Tiger Nim.

3.3 Sprague-Grundy (33 points)

Before introducing the incredible Sprague-Grundy Theorem, we present a lemma to prove the ubiquitous result.

Problem 3.3.1. (10 points) For every short impartial games G, G', we have $G = G'$ if and only if $G + G'$ is a P-position.

We now provide a very useful operation to help prove a lemma of the Sprague-Grundy Theorem and some later problems in the power round.

Definition 3.3.A. For a set $S \subset \mathbb{N} \cup \{0\}$, we define the *mex of* S *(minimal excluded value)*, denoted $\text{mex}(S)$, as the least integer $m \in \mathbb{N} \cup \{0\}$ such that $m \notin S$.

Problem 3.3.2. (8 points) Let $a_1, a_2, \ldots, a_k \in \mathbb{N} \cup \{0\}$, and suppose that $G = \{*a_1, *a_2, \ldots, *a_k \mid *a_1, *a_2, \ldots, *a_k\}$. Prove that $G = *m$, where $m = \text{mex}\{a_1, a_2, \ldots a_k\}$.

And now the main theorem.

Problem 3.3.3. (6 points) (Sprague-Grundy Theorem) Every short impartial game under the normal play convention is equivalent to a nimber.

The Sprague-Grundy Theorem is quite powerful in the analysis of short impartial games. In general, we can compare short impartial games with a nimber to understand its properties.

Definition 3.3.B. For a short impartial game G equivalent to the nimber $*n$, let the *G-value* of G be n.

Note the G-value is not equal to the game value.

Game Definition 3.3.I. *Dawson's Kayles* consists of at least one row of connected boxes. On a player's turn, they remove two adjacent boxes from a single row, possibly disconnecting it (and splitting it into two smaller rows), or remove one box of their choosing. The first person who cannot move loses.

Problem 3.3.4. (9 points) Find the G-values for a game of Dawson's Kayles consisting of one row of n boxes for each integer n from 0 to 17.

4 Specific Games & Questions (390 points)

4.1 Toads and Frogs (75 points)

Recall the game Toads and Frogs is presented as the first combinatorial game in the power round (Game 3.2.1).

Problem 4.1.1. (10 points) Find with proof the game value of
T F T F F □ F T □□ T T F F □ T F F □□ T T.

Problem 4.1.2. (20 points) Prove (T F)^{m}T □ (T F)^n has game value 2^{-n} for all $m, n \in \mathbb{N} \cup \{0\}$.

Problem 4.1.3. (20 points) Prove for every dyadic rational q, there exists a Toads and Frogs game with game value q.

Problem 4.1.4. (25 points) Prove no matter whether the left or right player moves first, the game T^n □ F □ F has game value 0 for all $n \geq 2$.

4.2 Partizan Splittles (90 points)

Game Definition 4.2.I. In Partizan Splittles, each position consists of a number of heaps of tokens, and with each move, a player removes s tokens from one heap and can optionally split the remaining heap into two heaps. Additionally, two sets of positive integers S_L and S_R are fixed in advance, and the amount of tokens s the Left player can remove on a turn must be in S_L, and for Right the amount must be in S_R.

Of course, depending on the sets S_L and S_R, this game can take on different values. We analyze a few of the scenarios below. In these scenarios, we use G_n to denote the game value of a Partizan Splittles game played on n heaps.

Problem 4.2.1. (5 points) Prove that if $S_L = \{1, a_1, a_2, \ldots, a_j\}$ and $S_R = \{1, b_1, b_2, \ldots, b_k\}$, where each a_i and each b_i is a positive odd integer, then

$$G_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ * & \text{if } n \text{ is odd} \end{cases}$$

Problem 4.2.2. (15 points) Prove that if $S_L = \{1\}$ and $S_R = \{k\}$, where k is a positive odd integer, then

$$G_n = \begin{cases} n & \text{if } n < k \\ \{k-1 \mid 0\} + G_{n-k} & \text{if } n \geq k \end{cases}$$

Problem 4.2.3. (50 points) Consider S_L and S_R with the properties that $1 \in S_L$ and $S_R = \{1, 3, 5, ..., 2k+1\}$ for some integer k or $S_R = \{1, 3, 5, ...\}$ (i.e. all odd integers).

a) Prove that $G_n \leq G_{n+2}$.

b) Prove that $G_{2n+1} = G_{2n} + *$.

c) Prove that if S_R is finite and $n - i - j \geq 2k$ is even then $G_n \leq G_i + G_j$.

Problem 4.2.4. (20 points) Let n be odd and let H_n be the game where $S_L = \{1\}$ and S_R is the set of even integers. Prove that $H_{n+1} - H_n < 1$ and $H_n - H_{n-1} = 1$.

4.3 Wythoff (225 points)

Game Definition 4.3.I. In *Wythoff*, there are two piles of coins. On a given turn, a player may either remove as many tokens from one pile as they wish or they may remove the same number of tokens from both piles. The winner is the one who removes the last coin.

For example, if we have a pile of 4 and 6 coins, then the first player can take away 3 from both piles leaving 1 and 3. The second player can then take away 3 from the second pile leaving just 1 coin. The first player then removes the last coin and wins.

Before we continue, we introduce an important and relevant result from number theory.

Problem 4.3.1. (10 points) Let $A = \{\lfloor nr \rfloor : n \in \mathbb{N}\}$ and $B = \{\lfloor ns \rfloor : n \in \mathbb{N}\}$ for some real numbers r and s. Prove $A \cup B = \mathbb{N}$ and $A \cap B = \emptyset$ if $r > 0$ and $s > 0$ are two irrational numbers such that $\frac{1}{r} + \frac{1}{s} = 1$.

Now, we have the main and intriguing result of the traditional Wythoff Game.

Problem 4.3.2. (20 points) Let $\phi = \frac{1+\sqrt{5}}{2}$. Prove that a P-position of Wythoff is a pair of piles with sizes given by the unordered pair $(\lfloor n\phi \rfloor, \lfloor n\phi^2 \rfloor)$ for some $n \in \mathbb{N}$.

We present further results involving G-values and Wythoff's Game.

Problem 4.3.3. (20 points) Prove that every G-value n appears exactly once among all ordered pairs of piles with the first pile of any given, fixed size.

Problem 4.3.4. (25 points) Let $r > 0$. r-Wythoff is played with two piles of tokens. On their turn, a player may either remove as many tokens from one pile as they wish, or remove a tokens from one pile and b from the other, where $|a - b| < r$. Prove that the n^{th} P-position of r-Wythoff is given by $(a_n, b_n) = (\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor)$ where $\alpha = \frac{1}{2}(2 - r + \sqrt{r^2 + 4})$ and, $\beta = \alpha + r$.

We give a nice theorem and useful definitions to solve the final problem.

Definition 4.3.A. Let T_j be the sequence of all pairs (a, b), $a \leq b$, of nonnegative integers such that the Wythoff game of those pile sizes has game value j. This sequence is of the form $\{(a_0, b_0), (a_1, b_1), \ldots\}$ where a_i is increasing (It actually can be proven (and used) that it is strictly increasing). Let $A_j = \{a_0, a_1, \ldots\}$ and similarly for B_j.

Definition 4.3.B. Let $d_1 \geq -1$ be an integer satisfying

$$\{j : 0 \leq j \leq d_1\} = \{b_j - a_j : 0 \leq j \leq d_1\}$$

The set of all pairs $(a_i, b_i) \in T_1$ for which there is an integer $d_2 > d_1$ satisfying

$$\{i : d_1 < i \leq d_2\} = \{b_i - a_i : d_1 < i \leq d_2\}$$

$$\{i : d_1 < i \leq d\} \neq \{b_i - a_i : d_1 < i \leq d\}$$

for every d with $d_1 < d < d_2$, is called an *integral*. The *size* of the above integral is $d_2 - d_1 + 1$.

Theorem 4.3.I. If

$$\{i : 0 \le i \le d\} = \{b_i - a_i : 0 \le i \le d\}$$

where $a_i \in A_1$ and $b_i \in B_1$, then the same holds for d replaced by $d + j$ for some $j \in \{1, 2, 3, 4, 5, 6\}$.

You can take this theorem for granted.

Definition 4.3.C. $B_1' = \{b_0', b_1', \ldots\}$ be the members of B_1 in sorted order (least to greatest).

Note: the last problem is very difficult, and we do not expect many teams to solve it. Grading will be similar to Olympiad grading, where scores will be clustered toward close to 0 points and close to full points.

Problem 4.3.5. (150 points) Let $\phi = \frac{1+\sqrt{5}}{2}$. Prove that

$$8 - 6\phi < a_n - \phi n < 6 - 3\phi$$

and

$$2 - 3\phi < b_n' - \phi^2 n < 6 - 3\phi$$

for $a_n \in A_1$ and $b_n' \in B_1'$.

This problem implies the pairs of piles of G-value 1 are really close to those of G-value 0, a cool result.

Team Number: ______________________________

PUMaC 2018 Power Round Cover Sheet

Remember that this sheet comes first in your stapled solutions. You should submit solutions for the problems in increasing order. Write on one side of the page only. The start of a solution to a problem should start on a new page. Please mark which questions for which you submitted a solution to help us keep track of your solutions.

Problem Number	Points	Attempted?
1.1.1	5	
1.1.2	5	
1.1.3	5	
1.1.4	6	
1.1.5	2	
1.1.6	4	
1.1.7	4	
1.1.8	6	
1.1.9	4	
1.2.1	4	
1.2.2	8	
1.2.3	13	
1.2.4	15	
1.2.5	12	
2.1.1	3	
2.2.1	10	
2.2.2	2	
2.2.3	4	
2.2.4	4	
2.2.5	2	
2.3.1	3	
2.3.2	6	
2.3.3	5	
2.3.4	2	
2.3.5	2	
2.3.6	4	
2.3.7	3	
2.3.8	5	
2.3.9	8	

Problem Number	Points	Attempted?
3.1.1	4	
3.1.2	6	
3.1.3	4	
3.1.4	4	
3.2.1	5	
3.2.2	8	
3.2.3	4	
3.2.4	5	
3.2.5	5	
3.2.6	12	
3.2.7	2	
3.3.1	10	
3.3.2	8	
3.3.3	6	
3.3.4	9	
4.1.1	10	
4.1.2	20	
4.1.3	20	
4.1.4	25	
4.2.1	5	
4.2.2	15	
4.2.3	50	
4.2.4	20	
4.3.1	10	
4.3.2	20	
4.3.3	20	
4.3.4	25	
4.3.5	150	
Total	638	