Number Theory A Solutions

1. Compute the last two digits of $9^{2020} + 9^{2020^2} + \ldots + 9^{2020^{2020}}$.

   Proposed by: Nancy Xu

   Answer: $\boxed{20}$

   It is enough to compute the residue of $9^{2020} + 9^{2020^2} + \ldots + 9^{2020^{2020}}$ modulo 100. We have:

$$\begin{aligned} 9^{2020} &\equiv (10-1)^{2020} \pmod{100} \\ &\equiv \sum_{n=0}^{2020} \binom{2020}{n} (10)^n (-1)^{2020-n} \pmod{100} \\ &\equiv \binom{2020}{1} (10)(-1)^{2019} + (-1)^{2020} \pmod{100} \\ &\equiv -20200 + 1 \pmod{100} \\ &\equiv 1 \pmod{100}. \end{aligned}$$

   Then $9^{2020^k} \equiv 1^k \pmod{100} \equiv 1 \pmod{100}$ for all k, so $9^{2020} + 9^{2020^2} + \ldots + 9^{2020^{2020}} \equiv 2020 \equiv 20 \pmod{100}$.

2. How many ordered triples of nonzero integers (a, b, c) satisfy $2abc = a + b + c + 4$?

   Proposed by: Austen Mazenko

   Answer: $\boxed{6}$

   Since $2ab - 1 \neq 0$ for integers a, b, we need $c = \frac{a+b+4}{2ab-1}$ to be an integer. If $|a|, |b| \geq 2$ then $|2ab - 1| > |a + b + 4|$ unless $a = b = 2$, so $c = \frac{8}{7}$. Thus, one of a, b is in $\{-1, 1\}$. If $a = 1$, then $(2b - 1)|(b + 5)$ and $b = 1, 6$, giving $(1, 1, 6)$ and cyclic permutations. If $a = -1$, then $(2b+1)|(b+3)$, so $b = -1$ or $b = 2$. In either case, we get $(-1, -1, 2)$ and cyclic permutations. This exhausts all possible cases, so our answer is 6.

3. Find the sum (in base 10) of the three greatest numbers less than 1000_{10} that are palindromes in both base 10 and base 5.

   Proposed by: Henry Erdman

   Answer: $\boxed{1584}$

   Noting that $2 \times 5^4 > 1000$, first we consider palindromes of the form $1XXX1_5$. Such numbers are greater than $5^4 = 625$. Note, however, that the final digit (in base 10) must be congruent to 1 modulo 5, so the greatest palindrome in both bases is of the form $6X6_{10}$. Thus we have ten options, and by trial and error, we find $676_{10} = 10201_5$ and $626_{10} = 10001_5$. These are the two largest numbers that satisfy our conditions, so we only have to find the next-largest. Note that any number greater than 4000_5 is also greater than 500_{10} and thus cannot be a palindrome in base 10 as well, since we have no number $500_{10} < x < 625_{10}$ such that the first and last digit match and are congruent to 4 modulo 5. Similarly, for $x > 3000_5$, we need $375_{10} < x < 500_{10}$ and the first and last digits of x to be congruent to 3 modulo 5. The only such palindromes are 383_{10} and 393_{10}, neither of which are palindromes in base 5. Moving down to the range $2000_5 = 250_{10} < x < 375_{10}$, $292_{10} = 2132_5$ is not a palindrome in base 5, but $282_{10} = 2112_5$ is, thus we have found our third number. Summing in base 10, $676 + 626 + 282 = 1584$.

4. Given two positive integers $a \neq b$, let $f(a, b)$ be the smallest integer that divides exactly one of a, b, but not both. Determine the number of pairs of positive integers (x, y), where $x \neq y$, $1 \leq x, y, \leq 100$ and $\gcd(f(x, y), \gcd(x, y)) = 2$.

Proposed by: Frank Lu

Answer: $\boxed{706}$

First, note that $f(x, y)$ is a power of a prime; for any n that divides x but not y, if it has at least two distinct prime factors, then we can write n as $p_1^{e_1}n'$, where p_1 doesn't divide n'. Then, if $p_1^{e_1}$ divides y, then n' can't divide into y, and $n' < n$. Thus, we see that $f(x, y) = 2^e$ for some exponent $e \geq 1$. Furthermore, we see that $2|x, 2|y$ by gcd. WLOG, suppose that $f(x, y)$ divides x, but not y. Then, note that the largest power of 2 in y is $e - 1$; otherwise, either it is divisible by 2^e or that 2^{e-1} is not a divisor of y. Furthermore, the largest power of 2 dividing x is larger than that of y, giving that $e \geq 2$. Hence, $y = 2y'$, y' odd, and $x = 4x'$, x' a positive integer. Note also that either both must be divisible by 3, or neither are, else $f(x, y) \leq 3$. We will proceed with casework.

- Case 1: x is not divisible by 3. Then, note that y' only has prime factors that are at least 5. We also know that $1 \leq y' \leq 50$, yielding $50 - \frac{50}{2} - \lfloor \frac{50}{3} \rfloor + \lfloor \frac{50}{6} \rfloor = 50 - 25 - 16 + 8 = 17$ possibilities for y'. For x', we have $25 - \lfloor \frac{25}{3} \rfloor = 25 - 8 = 17$ cases here, giving us a total of 289.
- Case 2: x is divisible by 3. Then, $y = 6y', x = 12x'$, and all we need is that y' is odd. This yields us that we have $\lfloor \frac{100}{12} \rfloor = 8$ choices for x' and, as we need $1 \leq y' \leq 16$, 8 choices for y'. This has 64 cases.

Thus, our answer is $2 * (289 + 64) = 2 * 353 = 706$.

5. We say that a positive integer n is *divable* if there exist positive integers $1 < a < b < n$ such that, if the base-a representation of n is $\sum_{i=0}^{k_1} a_i a^i$, and the base-b representation of n is $\sum_{i=0}^{k_2} b_i b^i$, then for all positive integers $c > b$, we have that $\sum_{i=0}^{k_2} b_i c^i$ divides $\sum_{i=0}^{k_1} a_i c^i$. Find the number of non-divable n such that $1 \leq n \leq 100$.

Proposed by: Frank Lu

Answer: $\boxed{27}$

First, note that if n can be written as pq, where $1 < p < q$ are positive integers, then note that the base $n-1$ representation of n is $1(n-1)+1$, and the base $q-1$ representation $p(q-1)+p$, and for $c > n-1$ we have that $((c-1)+1)|(p(c-1)+c)$. Thus, we only need to consider the positive integers that aren't primes or square of primes.

Also, for $p > 2$, we see that base $p - 1$ yields that p^2 gives $(p-1)^2 + 2(p-1) + 1$, and base $p^2 - 1$ yields $p^2 - 1 + 1$, so thus for $c \geq p^2 - 1$ we have that $(c + 1)|(c^2 + 2c + 1)$.

Now, given integer n and base-a, suppose that the base-a representation of n is $\sum_{i=0}^{k} a_i a^i$, let $p_{a,n}(x)$ be the polynomial $\sum_{i=0}^{k} a_i x^i$. Then, note that if we write $p_{a,n}(x)$ as $p_{b,n}(x)q(x)+r(x)$, where $r(x)$ has degree less than $p_{b,n}(x)$. But then note that for sufficiently large x, $p_{b,n}(x) > r(x)$.

But then, we see that if $r(x) \neq 0$, then we see that for each integer $x > n$ that $p_{b,n}(x)|r(x)$ implies that $r(x) = 0$ for all x sufficiently large. But then r is the zero polynomial, giving that $p_{b,n}(x)|p_{a,n}(x)$.

If $p_{b,n}(x)$ and $p_{a,n}(x)$ are the same degree, we see that the latter is a scalar multiple of the former, by say, c. But then we see that $c < p$ and we need $c|p$, contradiction.

Otherwise, note then that if the degree of $p_{a,n}(x)$ is d, note then $1 < p_{b,n}(a) < a^d \leq p_{a,n}(a) = n$, which means that n isn't prime, contradiction.

Thus, we see that the only non-divable numbers are primes, 4, and 1. For 4, we see the base representations 100_2 and 11_3, which is not possible.

We thus list out the numbers: $1, 2, 3, 4, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67$ $, 71, 73, 79, 83, 89, 97$, yielding our answer of 27.

6. Find the number of ordered pairs of integers (x, y) such that 2167 divides $3x^2 + 27y^2 + 2021$ with $0 \le x, y \le 2166$. *Proposed by: Aleksa Milojevic*

   Answer: $\boxed{2352}$

   First, we observe that $2167 = 11 \cdot 197$, and so by Chinese Remainder Theorem we just determine the number of ways to do this for $p = 11$ and $p = 197$.

   For $p = 11$, this reduces down to the congruence $3x^2 + 27y^2 \equiv 3 \pmod{11}$, or that $x^2 + 9y^2 \equiv 1 \pmod{11}$. Since 9 is a square, we see that we can write $z = 3y$ and solve $x^2 + z^2 \equiv 1 \pmod{11}$, and get the same number of solutions (since we can then find y again given z).

   As for $p = 197$, we get that $3x^2 + 27y^2 \equiv 51 \pmod{197}$, or that $x^2 + 9y^2 \equiv 17 \pmod{197}$, which we may again write as $x^2 + z^2 \equiv 17 \pmod{197}$. Notice, however, that $197 \equiv 1 \pmod{4}$, meaning that 17 is a quadratic residue of 197 if and only if 197 is one of 17, or that 10 is a square $\pmod{17}$. We can see that $10^8 \pmod{17} \equiv (-2)^4 \pmod{17} \equiv -1 \pmod{17}$, meaning that, in fact, 17 is a non-quadratic residue $\pmod{197}$.

   We now claim that the first equation has 12 solutions, and the second has 196. Here let $p = 197$ and $r = 17$. Let the number of solutions be N for $x^2 + z^2 \equiv r \pmod{p}$, where $r \neq 0$. Then, we have $N = \sum_{a+b=r} (1 + \left(\frac{a}{p}\right))(1 + \left(\frac{b}{p}\right))$. Thus $N = p + \sum_a \left(\frac{a}{p}\right) + \sum_b \left(\frac{b}{p}\right) + \sum_{a+b=r} \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$. The first two sums are easily seen to be 0. As for the third one, we consider the possibilities that we're allowed to have. First, suppose that a, b are both squares; notice then that, since $197 \equiv 1 \pmod{4}$, -1 is a square too, so we find the number of solutions to $(x - y)(x + y) = x^2 - y^2 \equiv r \pmod{197}$. Notice that, given $x - y \neq 0$, we can find $x + y$ and thus x, y. This yields us with 196 solutions. But considering the signs that are allowed, we see that we can negate x, y freely, and since 17 isn't a square modulo 196, but -1 is, we can't have either be 0, yielding us with $\frac{p-1}{4} = 49$ solutions here.

   Therefore, since we have $\frac{p+1}{2} = 99$ squares, we thus have 50 pairs where a is a square, b isn't, and so 50 where b is a square, a isn't, and therefore 48 where neither are squares. However, notice that we have two terms, namely those with $(0, 17)$ and $(17, 0)$ that we subtract because they contribute 0, not 1. But then notice that we get $49 + 48 - 50 - 50 + 2 = -1$. Therefore, we have that $N = p - 1$.

   We now run this argument for $p = 11$. Notice that we end up getting that, for $a + b = 1$, since -1 is a nonquadratic residue for 11, we see that the number where a is a square, b isn't is the number of solutions $x^2 - y^2 = 1$, where $y \neq 0$. We have in total 10 solutions for x, y, of which 2 have $y = 0$, and then we divide again by 4 to get 2 solutions in total. Thus, we have $6 - 2 = 4$ pairs where both are squares, 2 again with one but not the other, and 3 where both are not squares. This then evaluates to 3. But again, here we have the pairs $(0, 1)$ and $(1, 0)$ which contribute 0 each, not 1, so we subtract 2. Therefore, we see that we have $p + 4 - 2 - 2 + 3 - 2 = p + 1$ solutions here.

   Our answer is thus $12 \cdot 196 = 2352$.

7. Let $\phi(x, v)$ be the smallest positive integer n so that 2^v divides $x^n + 95$ if it exists, or 0 if no such positive integer exists. Determine $\sum_{i=0}^{255} \phi(i, 8)$.

   Proposed by: Frank Lu

   Answer: $\boxed{2704}$

   All equivalences here are $\mod 256$.

   First, we observe that $6561 + 95 \equiv 6656 = 256 * 26 \equiv 0$, and $6561 = 3^8$, so we can write the desired divisibility as $2^8 | x^n - 3^8$.

We now instead compute the number of i such that $\phi(i, 8) = n$ for each $n > 0$. Write $n = b2^a$, where b is odd.

First, we'll show that $a \le 3$ for there to be at least one solution.

By continuing squaring, we see that $(-95)^2 \equiv 65, 65^2 \equiv 129, 129^2 \equiv 1$, which means that $3^{64} \equiv 1$, but 3^{32} is not equivalent to 1. But note that $x^{64} - 1 \equiv 0$ for all odd x, since writing $x = 2y + 1$ yields that $x^{64} - 1 \equiv 128(y + 63y^2) \equiv 0$. Thus, $x^{b2^a} \equiv 3^8$, with $a > 3$, implies that $1 \equiv 3^{2^{9-a}}$, contradiction with $a > 3$.

Now, we know that $a \le 3$. Note that we expand out to get that we want x so that $(x^b - 3^{2^{3-a}})(x^b + 3^{2^{3-a}}) \dots (x^{2^{a-1}b} + 3^{2^2})$. Note that none of the terms other than the first 2 can contribute a power of 2 that is larger than 2, since these terms will be equivalent to $2 \mod 4$. Note also that at most one of the first two terms can be divisible by 4.

If $a > 0$, then either $x^b \equiv 3^{2^{3-a}} \mod 2^{8-a}$, or $x^b \equiv -3^{2^{3-a}} \mod 2^{8-a}$. If $a = 0$, this is just $x^b \equiv 3^8$.

But b is odd, so it has an inverse modulo any power of 2. Raising each of these equations to their appropriate powers yields a unique solution modulo 2^{8-a}.

Thus, the number of solutions for n is 1 if $a = 0$ and 2^{a+1} if $1 \le a \le 3$.

Now, say $x^m \equiv x^n \equiv 3^8$. Write $m = y2^a, n = z2^b$, with y, z odd. If $a \ne b$, WLOG $a < b$.

Then $x^{b-a} = 1$ gives that $x^{2^a(2^{b-a}y-z)} \equiv 1$. But $2^{b-a}y - z$ would be odd, so we can raise this to $2^{b-a}y - z$'s inverse modulo 64, giving $x^{2^a} \equiv 1$, which means that $x^{y2^a} = 3^8 \equiv 1$, a contradiction.

If $a = b$, repeating this yields that $x^{2^a(y-z)} \equiv 1$, or that $3^{8(y-z)}$, by raising to the yth power. But then we note that $y - z$ must be divisible by 8. Thus, we see that we have 16 possible values of $n : 1, 3, 5, 7, 2, 6, 10, 14, 4, 12, 20, 28, 8, 24, 40, 56$.

Summing these yields the answer $(1+3+5+7)(1*1+2*4+4*8+8*16) = 16*(1+8+32+128) = 16*(169) = 2704$.

8. What is the smallest integer a_0 such that, for every positive integer n, there exists a sequence of distinct positive integers $a_0, a_1, ..., a_{n-1}, a_n$ such that $a_0 = a_n$, and for $0 \le i \le n-1$, $a_i^{a_{i+1}}$ ends in the digits $\overline{0a_i}$ when expressed without leading zeros in base 10?

Proposed by: Austen Mazenko

Answer: $\boxed{7}$

Evidently, a_0 must be relatively prime to 10. First, we note that $a_0 \ne 3$; if it were, then $3^{a_1} \equiv 3 \pmod{100}$, and since $\text{ord}_{100}(3) = 20$ we need $a_1 \equiv 1 \pmod{20}$. Furthermore, if a_1 has k digits, we need $a_1^3 \equiv a_1 \pmod{10^{k+1}}$, so $(a_1-1)(a_1+1) \equiv 0 \pmod{10^{k+1}}$. Thus, $a_1 \equiv 1 \pmod{5^{k+1}}$, which combined with $a_1 \equiv 1 \pmod 4$ means $\nu_2(a_1+1) = 1$. But, $\nu_2((a_1-1)(a_1+1)) \ge k+1$ so $\nu_2(a_1 - 1) \ge k$. In particular, $a_1 - 1 \ge 2^k \cdot 5^{k+1} = 5 \cdot 10^k$, so a_1 has more than k digits, contradiction.

Now we claim that $a_0 = 7$ works. If $n = 2$, then pick $a_1 = 2 \cdot 5^8 - 1 = 781249$. First, $\text{ord}_{100}(7) = 4$, and since $781249 \equiv 1 \pmod 4$ we have $7^{781249} \equiv 7 \pmod{100}$. Then, $781249^2 \equiv 2 \cdot 5^8 \cdot (2 \cdot 5^8 - 2) + 1 \equiv 2^2 \cdot 5^8 \cdot (5^8 - 1) \equiv 1 \pmod{10^7}$ since $\nu_2(5^8 - 1) = 2 + 3 + 1 - 1 = 5$ by LTE. Hence, $781249^6 \equiv (781249^2)^3 \equiv 1 \pmod{10^7}$, as desired.

Otherwise, consider the arbitrarily long sequence $a_0 = 7, a_k = 2 \cdot 10^k + 1, a_{n-1} = 74218751$ for $0 < k < n-1$ First, $21 \equiv 1 \pmod 4$ implies $7^{21} \equiv 7 \pmod{100}$. Now, by the binomial theorem it is evident $(2\cdot 10^k+1)^{50} \equiv 1 \pmod{10^{k+2}}$, and because $2\cdot 10^{k+1}+1 \equiv 1 \pmod{50}$ for $k \ge 1$, we have $(2\cdot 10^k+1)^{2\cdot 10^{k+1}+1} \equiv 2\cdot 10^k+1 \pmod{10^{k+2}}$, and similarly for the exponent $74218751 \equiv 1 \pmod{50}$. It remains to show $74218751^2 \equiv 1 \pmod{10^9}$. We have $74218750 = 2 * 5^9 * 19 + 1$, so $74218751^2 - 1 = 2 \cdot 5^9 \cdot 19 \cdot 2(5^9 \cdot 19 + 1)$, meaning we must show $\nu_2(5^9 \cdot 19 + 1) \geq 7$. Now, $5^3 \equiv -3 \pmod{2^7}$, so $5^9 \equiv -27 \pmod{2^7}$, thus $5^9 \cdot 19 + 1 \equiv -27 \cdot 19 + 1 \equiv -512 \equiv 0 \pmod{2^7}$, as desired.