P U M :. C

Number Theory B Solutions

1. Andrew has a four-digit number whose last digit is 2. Given that this number is divisible by 9, determine the number of possible values for this number that Andrew could have.

   Proposed by: Frank Lu

   Answer: $\boxed{100}$

   It suffices to find the smallest and largest four-digit numbers that satisfy these conditions, because any two such numbers differ by a multiple of 90. We recall that an integer is divisible by 9 if and only if the sum of the digits is divisible by 9. So for the smallest integer, this is 1062. For the largest, we note that the largest possible sum of digits is 27. This yields the integer 9972.

   Their difference is 8910, and dividing by 90 yields 99. Hence, there are 100 such numbers.

2. The smallest three positive proper divisors of an integer n are $d_1 < d_2 < d_3$ so that $d_1+d_2+d_3 = 57$. Find the sum of the possible values of d_2.

   Proposed by: Frank Lu

   Answer: $\boxed{42}$

   Note that $d_1 = 1$ for all n. It suffices to solve $d_2 + d_3 = 56$.

   Note that the only possibilities that we have for d_2, d_3 are either that d_2, d_3 are distinct primes, or that d_3 is equal to d_2^2, where d_2 is a prime. For the second, notice that there is only one possible solution, namely by $d_2^2 + d_2 = 56$, yielding us with $d_2 = 7$.

   In the other case, we see that we want $d_2, 56-d_2$ to both be primes. We see that the primes less than 28 are $2, 3, 5, 7, 11, 13, 17, 19, 23$, with $56-d_2$ equaling, respectively, $54, 53, 51, 49, 45, 43, 39, 37, 33$. But from this list, the only pairs that work are $(3, 53), (13, 43)$, and $(19, 37)$. Therefore, it follows that the sum of the possible d_2 is $3 + 7 + 13 + 19 = 42$.

3. Compute the remainder when $2^{3^5} + 3^{5^2} + 5^{2^3}$ is divided by 30.

   Proposed by: Matthew Kendall

   Answer: $\boxed{6}$

   Computing the remainder modulo 2:

   $$2^{3^5} + 3^{5^2} + 5^{2^3} \equiv 0 + 1^{5^2} + 1^{2^3} \equiv 0 \pmod 2,$$

   modulo 3,

   $$2^{3^5} + 3^{5^2} + 5^{2^3} \equiv (-1)^{3^5} + 0 + (-1)^{2^3} \equiv 0, \pmod 3$$

   and modulo 5 using Fermat's Little Theorem,

   $$2^{3^5} + 3^{5^2} + 5^{2^3} \equiv 2^3 + 3^1 + 0 \equiv 1 \pmod 5.$$

   By Chinese Remainder, we know the remainder must be **6**.

4. A substring of a number n is a number formed by removing any number of digits from the beginning and end of n (not necessarily the same number of digits are removed from each side). Find the sum of all prime numbers p that have the property that any substring of p is also prime.

   Proposed by: Daniel Carter

   Answer: $\boxed{576}$

The prime numbers in question are 2, 3, 5, 7, 23, 37, 53, 73, and 373, which sum to 576. One can find the one- and two-digit primes with this property without much difficulty. Given those, the only candidate three-digit numbers are 237, 373, 537, and 737, of which only 373 is prime. Then one can see immediately that there are no four-digit primes with this property, since both the first and last three digits must also be primes with this property, i.e. they must both be 373. This also means there are no primes with five or more digits with this property.

5. Compute the number of nonnegative integral ordered pairs (x, y) such that $x^2 + y^2 = 32045$.

   Proposed by: Nancy Xu

   Answer: $\boxed{16}$

   We can write $32045 = 5 \cdot 13 \cdot 17 \cdot 29 = (1+2i)(1-2i)(2+3i)(2-3i)(1+4i)(1-4i)(2+5i)(2-5i)$, and from here we can write $x^2+y^2 = (x-yi)(x+yi) = 32045$ by taking the product of one of each of the conjugate pairs. There are 2 options for each conjugate pair for a total of $\frac{2^4}{2} = 8$ to account for overcounting, but x and y can be swapped, so there are 16 nonnegative ordered pairs.

6. Let $f(n) = \sum_{\gcd(k,n)=1, 1 \le k \le n} k^3$. If the prime factorization of $f(2020)$ can be written as $p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$, find $\sum_{i=1}^{k} p_i e_i$.

   Proposed by: Frank Lu

   Answer: $\boxed{818}$

   First, note that we can write $\sum_{i=1}^{n} i^3 = \sum_{d|n} \sum_{\gcd(i,n)=d} i^3 = \sum_{d|n} \sum_{\gcd(i/d,n/d)=1} d^3 i^3 = \sum_{d|n} d^3 f(n/d)$. But then we have that $(\frac{n^2+n}{2})^2 = \sum_{d|n} d^3 f(n/d)$. Now, note that, for a constant k dividing n, we have that $\sum_{k|d,d|n} d^3 f(n/d) = \sum_{k|d,d|n} (kd')^3 f(n/d) = k^3(\frac{(n/k)^2+(n/k)}{2})^2$. Then, we can use a PIE-esque argument based on divisibility by each of the prime factors (and products of these prime factors), yielding us, after simplifying, $\frac{n^2}{4}(p_1-1)\dots(p_k-1)(\frac{n^2}{p_1 \dots p_k}+(-1)^k)$. We thus find that $f(2020) = 2020^2/4*4*100*4039$, which equals $2^6*5^4*101^2*4039 = 2^6*5^4*7*101^2*577$, yielding us the answer of $12+20+7+202+577 = 32+786 = 818$.

7. Suppose that $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$, such that $f(x, y) = f(3x+y, 2x+2y)$. Determine the maximal number of distinct values of $f(x, y)$ for $1 \le x, y \le 100$.

   Proposed by: Frank Lu

   Answer: $\boxed{8983}$

   Note that the only places where we can get distinct values for $f(x, y)$ are those that are not of the form $(3a+b, 2a+2b)$ for some integers (a, b) in the range $1 \le a, b \le 100$. Observe that if $x = 3a+b, y = 2a+2b$, then we'd have that $a = \frac{2x-y}{4}, b = \frac{3y-2x}{4}$. In other words, for this to occur, we need that $2x \equiv y, 3y \pmod 4$. But then we have that y is even and x is the same parity of $y/2$.

   Furthermore, for the points that are of the above form, in order for $1 \le a, b \le 100$ as well, we need $4 \le 2x - y \le 400$ and $4 \le 3y - 2x \le 400$. From here, we see that for a given value of y, we have that $y + 4 \le 2x \le 3y - 4$, as the other two bounds are automatically satisfied as $1 \le x, y \le 100$. But then with $y = 2y_1$, we see that $y_1 + 2 \le x \le 3y_1 - 2$. For $y_1 \le 34$, we see that both bounds are the final bounds, meaning that, as x is the same sign as y_1, we have $y_1 - 1$ values for x. Over the values of y_1 this yields us with $33 \cdot 17 = 561$.

For $35 \leq y_1 \leq 50$, we have $y_1 + 2 \leq x \leq 100$ as the sharp bounds. Notice that this yields us with $\lfloor \frac{100-y_1}{2} \rfloor$ values for x, again maintaining the parity condition. Summing over these values yields us with $25 + 25 + 26 + 26 + \cdots + 32 + 32 = 57 \cdot 8 = 456$ values, so in total we have $561 + 456 = 1017$ values of (x, y) that are images of the function that sends (x, y) to $(3x + y, 2x + 2y)$ within $1 \leq x, y \leq 100$.

The number of distinct values of $f(x, y)$ is then at most $100^2 - 1017 = 8983$.

8. Let $f(n) = \sum_{i=1}^{n} \frac{\gcd(i,n)}{n}$. Find the sum of all n so that $f(n) = 6$.

Proposed by: Frank Lu

Answer: $\boxed{1192}$

Note that, the number of i so that $\gcd(i, n) = d$ is $\phi(n/d)$, if $n|d$. Then, we see that $f(n) = \sum_{i=1}^{n} \gcd(i, n) = \sum_{d|n} d\phi(n/d) = \sum_{d|n} n/d\phi(d)$ Now, suppose that n has prime factorization $n = p_1^{e_1} \ldots p_k^{e_k}$. Then, note that, since $\frac{1}{d}\phi(d)$ is multiplicative, we can write $f(n)/n$ as $\prod_{i=1}^{k} \sum_{j=0}^{e_i} \frac{1}{p_i^j} \phi(p_i^j) = \prod_{i=1}^{k} (1 + \sum_{j=1}^{e_i} \frac{1}{p_i^j} p_i^{j-1}(p-1)) = \prod_{i=1}^{k} (1 + \sum_{j=1}^{e_i} \frac{p_i - 1}{p_i}) = \prod_{i=1}^{k} (1 + \frac{e_i(p_i-1)}{p_i}). = \prod_{i=1}^{k} (\frac{(e_i+1)p_i - e_i}{p_i})$. Now, for this to be even, we need that the numerator of this product to first be even. But note that for p_i odd that $(e_i+1)p_i - e_i$ is odd, which means that one of our primes has to be 2, which say is $p_1 = 2$. Furthermore, we need that $e_1/2 + 1$ needs to be even for the product to equal 6. We thus see that $e_1 = 2, 6, 10$. For $e_1 = 10$, we see that we just have one prime factor, which means that we get the number $n = 2^{10} = 1024$. For $e_1 = 6$, we have that $e_1/2 + 1 = 4$, which is too small. However, note also that the smallest possible value for any other term in the product, with $p_i \geq 3$, is $5/3 > 3/2$. For $e_1 = 2$, we have that $e_1/2 + 1 = 2$, again is too small. We want the product of the next terms to be 3. Note that we can't have more than 2 other prime factors, the product of this is at most $5/3 \cdot 9/5 \cdot 13/7 = 39/7 > 3$. For 2 prime factors, the smallest possible value of the terms due to the other factors is $5/3 \cdot 9/3 = 3$, giving $n = 2^2 \cdot 3 \cdot 5 = 60$. For 1 prime factor, we want $1 + \frac{e_2(p_2-1)}{p_2} = 3$, or that $e_2(p_2 - 1) = 2p_2$, which requires $p_2|e_2$, or $p_2 - 1|2$, meaning that $p_2 = 3$, and that $e_2 = 3$. This gives $n = 2^2 \cdot 3^3 = 108$. Our total sum is thus $108 + 1024 + 60 = 1192$.