P U M ∴ C

Team Round

1. Let $T = \{a_1, a_2, \ldots, a_{1000}\}$, where $a_1 < a_2 < \ldots < a_{1000}$, be a uniformly randomly selected subset of $\{1, 2, \ldots, 2018\}$ with cardinality 1000. The expected value of a_7 can be written in reduced form as $\frac{m}{n}$. Find $m + n$.

   Proposed by: Jackson Blitz

   Answer: $\boxed{2162}$

   There is a bijection between subsets $\{a_1, \ldots, a_{1000}\}$ of $\{1, \ldots, 2018\}$ (with $a_1 < \cdots < a_{1000}$) and sequences $(b_1, \ldots, b_{1001})$ of positive integers adding to 2019, with the bijection given by $a_n = a_{n-1} + b_n$ (with $a_0 = 0$ and $a_{1001} = 2019$). Then $a_7 = b_1 + b_2 + \cdots + b_7$, so the expected value of a_2 is 7 times the expected value of any b_i (by symmetry every b_i has the same expected value, and therefore this expected value is $\frac{2019}{1001}$). Thus, the expected value of a_2 is $\frac{7 \cdot 2019}{1001} = \frac{2019}{143}$, giving an answer of $\boxed{2162}$.

2. Let triangle $\triangle ABC$ have $AB = 90$ and $AC = 66$. Suppose that the line IG is perpendicular to side BC, where I and G are the incenter and centroid, respectively. Find the length of BC.

   Proposed by: Jackson Blitz

   Answer: $\boxed{52}$

   Let H be the foot of the altitude from A to BC, let M be the midpoint of BC, and let D be the intersection of IG and BC. Let $BM = x$. Since $AB > AC$, points B, M, D, C are in that order along the line, and we have $BD - DC = BA - AC = 24$, so that $BD = x + 12$ and $MD = 12$. Because triangles GMD and AMH are similar with a similarity ratio of 3, we have that $MH = 36$. Finally, this implies that $BH = x + 36$ and $CH = |x - 36|$, and we have that $BH^2 - CH^2 = BA^2 - CA^2$. Solving this, we have that $x = 26$, so the answer is $BC = 52$.

3. The value of
$$\frac{\log_3 5 \log_2 5}{\log_3 5 + \log_2 5}$$
   can be expressed as $a \log_b c$, where a, b, and c are positive integers, and $a + b$ is as small as possible. Find $a + 2b + 3c$.

   Proposed by: Michael Gintz

   Answer: $\boxed{28}$

   Using the fact that $\log_a b = \frac{\log_c b}{\log_c a}$, we get that this is equal to
$$\frac{\log_3 5 \log_2 5}{\left(\frac{\log_5 5(\log_5 3 + \log_5 2)}{\log_5 3 \log_5 2}\right)}$$
   Since $\log_a b \log_b a = 1$, this is
$$\frac{\left(\frac{1}{\log_5 3 \log_5 2}\right)}{\left(\frac{\log_5 5(\log_5 3 + \log_5 2)}{\log_5 3 \log_5 2}\right)} = \frac{1}{\log_5 3 + \log_5 2} = \frac{1}{\log_5 6} = \log_6 5$$
   Thus 1+12+15=28.

4. For how many positive integers n less than 2018 does n^2 have the same remainder when divided by 7, 11, and 13?

   Proposed by: Eric Neyman

Answer: 36

The quadratic residues mod 7 are 1, 2, and 4. Of these 1 and 4 are clearly quadratic residues mod 11 and 13, and it is easy to check that 2 is not a quadratic residue mod 11. So these numbers n are precisely 0 (mod $7 \cdot 11 \cdot 13$); ± 1 (possible different signs) modulo 7, 11, and 13; and ± 2. By the Chinese remainder theorem, there are exactly $1 + 8 + 8 = 17$ of these for every consecutive $7 \cdot 11 \cdot 13 = 1001$ numbers, so there are 34 of these through 2002. It is easy to check that 2003 and 2006 work (being 1 and 4, respectively, modulo the three primes), and the rest don't satisfy the above criteria. This gives an answer of 36.

5. There exist real numbers a, b, c, d, and e such that for all positive integers n, we have

$$\sqrt{n} = \sum_{i=0}^{n-1} \sqrt[5]{\sqrt{ai^5 + bi^4 + ci^3 + di^2 + ei + 1} - \sqrt{ai^5 + bi^4 + ci^3 + di^2 + ei}}.$$

Find $a + b + c + d$.

Proposed by: Jackson Blitz

Answer: 1656

We have that each summand should be $\sqrt{i+1} - \sqrt{i}$. We thus have

$$(\sqrt{n+1} - \sqrt{n})^5 = \sqrt{an^5 + bn^4 + cn^3 + dn^2 + en + 1} - \sqrt{an^5 + bn^4 + cn^3 + dn^2 + en}$$

Expanding, we have $\sqrt{an^5 + bn^4 + cn^3 + dn^2 + en} = (16n^2 + 20n + 5)\sqrt{n}$. Plugging in $n = 1$, we get $a + b + c + d + e = 1681$, and plugging in $n = 0$, we get $e = 25$, so the answer is $1681 - 25 = 1656$.

6. Let $\tau(n)$ be the number of distinct positive divisors of n (including 1 and itself). Find the sum of all positive integers n satisfying $n = \tau(n)^3$.

Proposed by: Jackson Blitz

Answer: 85953

The solutions are $n = 1^3, 28^3, 40^3$. The equation gives n is a cube. Let $k = \sqrt[3]{n}$. Let $k = p_1^{a_1} p_2^{a_2} \cdots$ where p_i are the primes in order. $\tau(n) = (3a_1 + 1)(3a_2 + 1) \cdots$. Hence $a_2 = 0$. When $p_i \geq 5$ and $a_i \neq 0$ we have $p_i^{a_i} > 3a_i + 1$. We also have $2^{a_1} > 3a_1 + 1$ for $a_1 \geq 4$. For later, we have $4 \cdot 5^{a_3 - 1} > 3(a_3 - 1) + 4$ for $a_3 \geq 2$ and similarly $4 \cdot 7^{a_4 - 1} > 3(a_4 - 1) + 4$ for $a_4 \geq 2$. This all gives $a_1 = 0, 1, 2, 3$. If $a_1 = 0$ then we easily have $a_3 = a_4 = \cdots = 0$ so $k = 1$. If $a_1 = 1$ then $2 \cdot 5^{a_3} \cdots = 4(3a_3 + 1) \cdots$. This is a contradiction so no solutions for k when $a_1 = 1$. If $a_1 = 2$ then $4 \cdot 5^{a_3} \cdots = 7(3a_3 + 1) \cdots$ so $a_4 \geq 1$. Then $4 \cdot 7^{a_4 - 1} 5^{a_3} \cdots = (3(a_4 - 1) + 4)(3a_3 + 1) \cdots$. We have equality when $a_3 = a_4 - 1 = \cdots = 0$ so $k = 2^2 \cdot 7 = 28$ is a solution. Furthermore from above we have there are no further solutions. A similar casework to $a_1 = 2$ gives only $k = 2^3 \cdot 5 = 40$ as the only solution when $a_1 = 3$. We have then exhausted all cases. $k = 1, 28, 40$ implies $n = 1^3, 28^3, 40^3$. The sum of these solutions is 85953.

7. Let triangle $\triangle MNP$ have side lengths $MN = 13, NP = 89$, and $PM = 100$. Define points S, R, B as midpoints of MN, NP, PM respectively. A line ℓ cuts lines MN, NP, PM in points I, J, A, respectively. Find the minimum value of $(SI + RJ + BA)^2$.

Proposed by: Marko Medvedev

Answer: 1157

Let $a = MN, b = NP, c = PM$. $SI + RJ + BA \geq \sqrt{ab}$
Let $\frac{\vec{SI}}{\vec{MN}} = \frac{1}{2x}, \frac{\vec{RJ}}{\vec{NP}} = \frac{1}{2y}, \frac{\vec{BA}}{\vec{PM}} = \frac{1}{2z}$. Now we have that $SI + RJ + BA = \frac{1}{2}|x|a + \frac{1}{2}|y|b + \frac{1}{2}|z|c \geq \sqrt{|x|(|y| + |z|)}\sqrt{ab}, \frac{1}{2}|x|a + \frac{1}{2}|y|b + \frac{1}{2}|z|c \geq \sqrt{|y|(|x| + |z|)}\sqrt{ab}, \frac{1}{2}|x|a + \frac{1}{2}|y|b + \frac{1}{2}|z|c \geq$

$\sqrt{|z|(|x|+|y|)}\sqrt{ab}$. It suffices to prove that at least one of the numbers $|x|(|y|+|z|), |y|(|z|+|x|), |z|(|x|+|y|)$ is at least 1. But this follows from the Menelaus theorem which says that $xy+xz+yz=-1$, and we know that at least one of the numbers xy, yz, xz is nonnegative, hence the result follows. In order to get equality, let l be the line containing B and choose $|x|=\sqrt{\frac{a}{b}}, |y|=\sqrt{\frac{b}{a}}$. Hence the minimum value is $a \cdot b$, hence the answer is $\boxed{1157}$.

8. Jackson has a 5×5 grid of squares. He places coins on the grid squares — at most one per square — so that no row, column, or diagonal has five coins. What is the maximum number of coins that he can place?

   Proposed by: Zhuo Qun Song

   Answer: $\boxed{20}$

   Jackson could place the first 20 coins onto the board such that one diagonal is fully omitted. However, if he placed 21 coins onto the board, there would exist at least one row where all five entries have a coin on them, so he must necessarily stop after placing the 20^{th} coin.

9. There are numerous sets of 17 distinct positive integers that sum to 2018, such that each integer has the same sum of digits in base 10. Let M be the maximum possible integer that could exist in any such set. Find the sum of M and the number of such sets that contain M.

   Proposed by: Jackson Blitz

   Answer: $\boxed{521}$

   We get that the least integer is at most $\frac{2018}{17} < 198$ so has sum of digits less than or equal to 18. Furthermore the 17 integers have sum mod 9 of $2018 \equiv 2 \mod 9$ implying they each are $\frac{2}{17} \equiv 7 \mod 9$ since a number equals its digit sum mod 9. Hence, each integer has digit sum either 7 or 16. If it were 16, then the possible integers would be $79, 88, 97, 169, 178, 187, 196, \ldots$ which quickly can bee seen to produce a sum greater than 2018 taking the smallest 17 elements. Hence the sum of digits of each of the 17 integers must be 7. The list of possible integers is Taking the smallest 17 integers with digit sum 7 $\{7, 16, 25, 34, 43, 52, 61, 70, 106, 115, 124, 133, 142, 151, 160, 205, 214\}$, we have a sum of 1658. Hence there is only $2018 - 1658 = 360$ of "wiggle room." We can quickly see the desired largest element is 520 and 601 is impossible to include as it is too high and the set $\{7, 16, 25, 34, 43, 52, 61, 70, 106, 115, 124, 133, 142, 151, 205, 214, 520\}$ satisfies. Now, it is quick to notice that this is in fact the only set that contains 520. The final answer is then $520+1=521$.

10. For how many ordered quadruplets (a, b, c, d) of positive integers such that $2 \le a \le b \le c$ and $1 \le d \le 418$ do we have that $bcd + abd + acd = abc + abcd$?

    Answer: $\boxed{420}$

    Write as

    $$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{d}+1$$

    If $a=3$ then the left side is too small. So $a=2$. Then we have

    $$\frac{1}{b}+\frac{1}{c}=\frac{1}{d}+\frac{1}{2}$$

    If $b=4$ then the left side is too small. So $b=2$ or $b=3$. If $b=2$ then $c=d$, giving 417 solutions.
    If $b=3$ then by SFFT we get $36=(6-c)(d+6)$ which has solutions $(c,d)=(3,6),(4,12),(5,30)$ giving 3 solutions.
    In total this gives $417+3=420$ solutions.

11. Let $\frac{a}{b}$ be a fraction such that a and b are positive integers and the first three digits of its decimal expansion are 0.527. What is the smallest possible value of $a+b$?

Proposed by: Nathan Bergman

Answer: 55

We know that $\frac{527}{1000} \leq \frac{a}{b} < \frac{528}{1000}$ so $527b \leq 1000a < 528b$. So we want $527b$ and $528b$ to have different thousandths places. If b is even, then let $b = 2c$ so that $527b = 527(2c) = 1000c + 54c$ and similarly $528b = 1000c + 56c$, and we can ignore the $1000c$ term. In this case, it is easy to see that the smallest value of c that works is 18, because $54 * 18 = 972$ and $56 * 18 = 1,008$. Then $b = 2c = 36$, and the corresponding value of a is 19. Now, we must consider the case if b is odd. Then we have a similar situation where we consider a value d such that $28d$ has a value greater than 500 but $27d$ is less than 500. In this case, we get that $d = 18$ once again, but then the smallest odd value must be larger than the smallest even value. So the smallest value of $\frac{a}{b}$ is $\frac{19}{36}$, so the answer is 55.

12. In right triangle $\triangle ABC$, a square $WXYZ$ is inscribed such that vertices W and X lie on hypotenuse AB, vertex Y lies on leg BC, and vertex Z lies on leg CA. Let AY and BZ intersect at some point P. If the length of each side of square $WXYZ$ is 4, the length of the hypotenuse AB is 60, and the distance between point P and point G, where G denotes the centroid of the triangle, is $\frac{a}{b}$, compute the value of $a+b$.

Proposed by: Henry Wang

Answer: 167

Let $BC = a$, $CA = b$, $AB = c$, and $WX = s$. Then, by similar triangles, $CY = a\frac{s}{c}$ and $CZ = b\frac{s}{c}$. By Ceva's Theorem, CP must therefore be the median. Hence, by mass points, we know that the ratio of CP to CM, where M is the midpoint of hypotenuse AB must be $\frac{2(\frac{s}{c})}{\frac{s}{c}+1}$. Because triangle ABC is right, $CM = AM = BM = \frac{c}{2}$ and hence $CM = \frac{2}{3}\frac{c}{2} = \frac{c}{3}$. Hence, the distance between P and G is $|\frac{c}{3} - \frac{2s}{s+c}|$ which is $|\frac{60}{3} - \frac{2(4)}{4+60}| = \frac{159}{8}$ and the desired answer is $159 + 8 = \boxed{167}$.

13. Consider a 10-dimensional $10 \times 10 \times \ldots \times 10$ cube consisting of 10^{10} unit cubes, such that one cube A is centered at the origin, and one cube B is centered at $(9,9,9,9,9,9,9,9,9,9)$. Paint A red and remove B, leaving an empty space. Let a move consist of taking a cube adjacent to the empty space and placing it into the empty space, leaving the space originally contained by the cube empty. What is the minimum number of moves required to result in a configuration where the cube centered at $(9,9,9,9,9,9,9,9,9,9)$ is red?

Proposed by: Michael Gintz

Answer: 357

Consider the "movement" of the empty space. Let us consider the minimum time between moving the red cube in the right direction. We know that after moving the red cube in the right direction, the empty space has just "switched places" with it so it is one step in the "negative" direction. So our coordinates if we call the red cube the origin might look like -1,0,0,0... To get the cube to move in the right direction, we need 1,0,0,0... up to permutation. We must do this in at least 2 moves, so you can only move the red cube the right way at most every 3 moves. We claim this is sufficient. Say there is any direction which our red cube has not gotten to the end of, say the second coordinate. We move from -1 0 0 0.. to -1 1 0 0.. to 0 1 0 0 0 to 0 0 0 0 0. Thus we can always make another step in 3 moves. It takes 90 moves for the empty cube to make its first move, and the red cube needs to take 90 steps, so the number of steps needed is $90 + 89 \cdot 3 = 357$.

14. Find the sum of the positive integer solutions to the equation $\lfloor \sqrt[3]{x} \rfloor + \lfloor \sqrt[4]{x} \rfloor = 4$.

Proposed by: Jackson Blitz

Answer: 231

Note $x > 1$. Since $\sqrt[3]{x} > \sqrt[4]{x}$ for $x > 1$ we have 3 cases: $\lfloor \sqrt[3]{x} \rfloor = 4, 3, 2$. If $\lfloor \sqrt[3]{x} \rfloor = 4$ then $x \geq 4^3 = 64$ but $\lfloor \sqrt[4]{x} \rfloor = 0$ so $x < 1^4 = 1$. So the first case is out. If $\lfloor \sqrt[3]{x} \rfloor = 3$ then $x \geq 3^3 = 27$ but $\lfloor \sqrt[4]{x} \rfloor = 1$ so $x < 2^4 = 16$. So then it must be the case that $\lfloor \sqrt[3]{x} \rfloor = \lfloor \sqrt[4]{x} \rfloor = 2$. This gives $3^3 > x \geq 2^3$ and $3^4 > x \geq 2^4$ meaning $x \in \{16, 17, \ldots, 26\}$. Thus, the answer is $16 + 17 + \cdots + 26 = \boxed{231}$.

15. Aaron the Ant is somewhere on the exterior of a hollow cube of side length 2 inches, and Fred the Flea is on the inside, at one of the vertices. At some instant, Fred flies in a straight line towards the opposite vertex, and simultaneously Aaron begins crawling on the exterior of the cube towards that same vertex. Fred moves at $\sqrt{3}$ inches per second and Aaron moves at $\sqrt{2}$ inches per second. If Aaron arrives before Fred, the area of the surface on the cube from which Aaron could have started can be written as $a\pi + \sqrt{b} + c$ where a, b, and c are integers. Find $a + b + c$.

Proposed by: Zachary Stier

Answer: 106

Consider unfolding the cube as in the first diagram. The lightest square is the bottom of the cube, and A is the vertex to which Aaron and Fred are going. The darkest squares are duplicates of the top. Fred takes 2 seconds to arrive, so Aaron must start from within a $2\sqrt{2}$ inch radius. We wish to compute the area on the top that Aaron can reach. This is the sum of the areas of the intersections of the circle with the two darkest squares, subtracted by the excess area that they intersect in.

That excess area can be found as a part of the lune in the second figure. The lune has area $\frac{16}{3}\pi - 2\sqrt{3}$ since the triangle drawn in the third figure has radius $2\sqrt{2}$ and vertex angle 120°. Thus, dividing by two and going back to the second figure, subtracting off the two arc of angle measure 45 and adding back the isosceles right triangle of leg 1 gives the remaining area as what we want, and it's equal to $\frac{2}{3}\pi - 2\sqrt{3} + 2$.

Thus, the total area of the top that Aaron can reach is equal to $2(\pi - 2) - (\frac{2}{3}\pi - \sqrt{3} + 2) = \frac{4}{3}\pi + 2\sqrt{3} - 6$. Finally, there are three such faces and Aaron can also be anywhere in the other three faces, so the total area he can be in is $4\pi + 6\sqrt{3} - 6$. The answer is thus 106.

16. Let N be the number of subsets B of the set $\{1, 2, \cdots, 2018\}$ such that the sum of the elements of B is congruent to 2018 modulo 2048. Find the remainder when N is divided by 1000.

Proposed by: Atanas Dinev

Answer: 128

Observe that subsets of $\{1, 2, 4, \ldots, 1024\}$ cover all residues modulo 2048. So for any subset of $\{3, 5, 6, 7, 9, 10, \ldots, 2018\}$ there is a unique subset of $\{1, 2, 4, \ldots, 1024\}$ that supplements it to be 2018 modulo 2048. Thus, the answer is the number of subsets of $\{3, 5, 6, \ldots, 2018\}$, i.e. 2^{2007}, modulo 1000. Observe that $2^{100} \equiv 1 \pmod{125}$ (by the Euler-Fermat theorem, since $100 = \varphi(125)$), so $2^{2000} \equiv 1 \pmod{125}$, so $2^{2007} \equiv 128 \pmod{125}$. Also, $2^{2007} \equiv 128 \pmod{8}$, so $2^{2007} \equiv \boxed{128} \pmod{100}$.

If you believe that any of these answers is incorrect, or that a problem had multiple reasonable interpretations or was incorrectly stated, you may appeal at http://tinyurl.com/PUMaCappeal2018. All appeals must be in by 1 PM to be considered.