P U M . : . C

# Team Round

The Team Round consists of 15 questions. Your team has **40 minutes** to complete the Team Round. Each problem is worth 5 points. At the conclusion of the test, in addition to providing answers to the problems, you will be asked to input an integer $I$. The *value* of your input is defined as $V = I - S$, where $S$ is the number of problems you solve correctly. If $V \leq 0$, you receive zero bonus points. If $V > 0$, the number of bonus points you receive will be $\frac{20.23}{2A+V}$, where $A$ is the number of teams (including your team) that have the same value $V$ as you. Good luck!

1. Have $b, c \in \mathbb{R}$ satisfy $b \in (0, 1)$ and $c > 0$, then let $A, B$ denote the points of intersection of the line $y = bx + c$ with $y = |x|$, and let $O$ denote the origin of $\mathbb{R}^2$. Let $f(b, c)$ denote the area of triangle $\triangle OAB$. Let $k_0 = \frac{1}{2022}$, and for $n \geq 1$ let $k_n = k_{n-1}^2$. If the sum $\sum_{n=1}^{\infty} f(k_n, k_{n-1})$ can be written as $\frac{p}{q}$ for relatively prime positive integers $p, q$, find the remainder when $p + q$ is divided by 1000.

2. A triangle $\triangle A_0A_1A_2$ in the plane has sidelengths $A_0A_1 = 7, A_1A_2 = 8, A_2A_0 = 9$. For $i \geq 0$, given $\triangle A_iA_{i+1}A_{i+2}$, let $A_{i+3}$ be the midpoint of $A_iA_{i+1}$ and let $G_i$ be the centroid of $\triangle A_iA_{i+1}A_{i+2}$. Let point $G$ be the limit of the sequence of points $\{G_i\}_{i=0}^{\infty}$. If the distance between $G$ and $G_0$ can be written as $\frac{a\sqrt{b}}{c}$, where $a, b, c$ are positive integers such that $a$ and $c$ are relatively prime and $b$ is not divisible by the square of any prime, find $a^2 + b^2 + c^2$.

3. Provided that $\{\alpha_i\}_{i=1}^{28}$ are the 28 distinct roots of $29x^{28} + 28x^{27} + \ldots + 2x + 1 = 0$, then the absolute value of $\sum_{i=1}^{28} \frac{1}{(1-\alpha_i)^2}$ can be written as $\frac{p}{q}$ for relatively prime positive integers $p, q$. Find $p + q$.

4. Patty is standing on a line of planks playing a game. Define a block to be a sequence of adjacent planks, such that both ends are not adjacent to any planks. Every minute, a plank chosen uniformly at random from the block that Patty is standing on disappears, and if Patty is standing on the plank, the game is over. Otherwise, Patty moves to a plank chosen uniformly at random within the block she is in; note that she could end up at the same plank from which she started. If the line of planks begins with $n$ planks, then for sufficiently large $n$, the expected number of minutes Patty lasts until the game ends (where the first plank disappears a minute after the game starts) can be written as $P(1/n)f(n) + Q(1/n)$, where $P, Q$ are polynomials and $f(n) = \sum_{i=1}^{n} \frac{1}{i}$. Find $P(2023) + Q(2023)$.

5. You're given the complex number $\omega = e^{2i\pi/13} + e^{10i\pi/13} + e^{16i\pi/13} + e^{24i\pi/13}$, and told it's a root of a unique monic cubic $x^3 + ax^2 + bx + c$, where $a, b, c$ are integers. Determine the value of $a^2 + b^2 + c^2$.

6. A sequence of integers $x_1, x_2, ...$ is *double-dipped* if $x_{n+2} = ax_{n+1} + bx_n$ for all $n \geq 1$ and some fixed integers $a, b$. Ri begins to form a sequence by randomly picking three integers from the set $\{1, 2, ..., 12\}$, with replacement. It is known that if Ri adds a term by picking another element at random from $\{1, 2, ..., 12\}$, there is at least a $\frac{1}{3}$ chance that his resulting four-term sequence forms the beginning of a double-dipped sequence. Given this, how many distinct three-term sequences could Ri have picked to begin with?

7. Pick $x, y, z$ to be real numbers satisfying $(-x+y+z)^2 - \frac{1}{3} = 4(y-z)^2$, $(x-y+z)^2 - \frac{1}{4} = 4(z-x)^2$, and $(x+y-z)^2 - \frac{1}{5} = 4(x-y)^2$. If the value of $xy + yz + zx$ can be written as $\frac{p}{q}$ for relatively prime positive integers $p, q$, find $p + q$.

8. Ryan Alweiss storms into the Fine Hall common room with a gigantic eraser and erases all integers $n$ in the interval $[2, 728]$ such that $3^t$ doesn't divide $n!$, where $t = \lceil \frac{n-3}{2} \rceil$.
Find the sum of the leftover integers in that interval modulo 1000.

9. In the complex plane, let $z_1, z_2, z_3$ be the roots of the polynomial $p(x) = x^3 - ax^2 + bx - ab$. Find the number of integers $n$ between 1 and 500 inclusive that are expressible as $z_1^4 + z_2^4 + z_3^4$ for some choice of positive integers $a, b$.

10. Let $\alpha, \beta, \gamma \in \mathbb{C}$ be the roots of the polynomial $x^3 - 3x^2 + 3x + 7$. For any complex number $z$, let $f(z)$ be defined as follows:
$$f(z) = |z - \alpha| + |z - \beta| + |z - \gamma| - 2 \max_{w \in \{\alpha, \beta, \gamma\}} |z - w|.$$
Let $A$ be the area of the region bounded by the locus of all $z \in \mathbb{C}$ at which $f(z)$ attains its global minimum. Find $\lfloor A \rfloor$.

11. For the function
$$g(a) = \max_{x \in \mathbb{R}} \left\{ \cos x + \cos\left(x + \frac{\pi}{6}\right) + \cos\left(x + \frac{\pi}{4}\right) + \cos(x + a) \right\},$$
let $b \in \mathbb{R}$ be the input that maximizes $g$. If $\cos^2 b = \frac{m+\sqrt{n}+\sqrt{p}-\sqrt{q}}{24}$ for positive integers $m, n, p, q$, find $m + n + p + q$.

12. Observe the set $S = \{(x, y) \in \mathbb{Z}^2 : |x| \leq 5 \text{ and } -10 \leq y \leq 0\}$. Find the number of points $P$ in $S$ such that there exists a tangent line from $P$ to the parabola $y = x^2 + 1$ that can be written in the form $y = mx + b$, where $m$ and $b$ are integers.

13. Of all functions $h : \mathbb{Z}_{>0} \to \mathbb{Z}_{\geq 0}$, choose one satisfying $h(ab) = ah(b) + bh(a)$ for all $a, b \in \mathbb{Z}_{>0}$ and $h(p) = p$ for all prime numbers $p$. Find the sum of all positive integers $n \leq 100$ such that $h(n) = 4n$.

14. Let $\triangle ABC$ be a triangle. Let $Q$ be a point in the interior of $\triangle ABC$, and let $X, Y, Z$ denote the feet of the altitudes from $Q$ to sides $BC$, $CA$, $AB$, respectively. Suppose that $BC = 15$, $\angle ABC = 60^\circ$, $BZ = 8$, $ZQ = 6$, and $\angle QCA = 30^\circ$. Let line $QX$ intersect the circumcircle of $\triangle XYZ$ at the point $W \neq X$. If the ratio $\frac{WY}{WZ}$ can be expressed as $\frac{p}{q}$ for relatively prime positive integers $p, q$, find $p + q$.

15. Subsets $S$ of the first 35 positive integers $\{1, 2, 3, ..., 35\}$ are called *contrived* if $S$ has size 4 and the sum of the squares of the elements of $S$ is divisible by 7. Find the number of contrived sets.

**Team:**

**Write answers in table below:**

| Q1 | Q2 | Q3 | Q4 | Q5 | Q6 | Q7 | Q8 |
|---|---|---|---|---|---|---|---|
| | | | | | | | |

| Q9 | Q10 | Q11 | Q12 | Q13 | Q14 | Q15 |
|---|---|---|---|---|---|---|
| | | | | | | |

**Your input $I$ (for bonus points):**