Team Round Solutions

1. Given $n \geq 1$, let A_n denote the set of the first n positive integers. We say that a bijection $f : A_n \to A_n$ has a *hump* at $m \in A_n \setminus \{1, n\}$ if $f(m) > f(m+1)$ and $f(m) > f(m-1)$. We say that f has a hump at 1 if $f(1) > f(2)$, and f has a hump at n if $f(n) > f(n-1)$. Let P_n be the probability that a bijection $f : A_n \to A_n$, when selected uniformly at random, has exactly one hump. For how many positive integers $n \leq 2020$ is P_n expressible as a unit fraction?

   Proposed by Oliver Thakar

   Answer: $\boxed{11}$

   Fix n. Let $N(n, k)$ be the number of bijections $f : A_n \to A_n$, that has one hump at k, and no others. (Then, notice that $f(k) = n$.) I claim that $N(n, k) = \binom{n-1}{k-1}$.

   I prove this claim by induction on n. For the base case, when $n = 1$, we have $N(1, 1) = 1 = \binom{0}{0}$.

   Otherwise, assume that $N(n-1, k) = \binom{n-2}{k-1}$ for all k. Then, notice that a bjiection $f : A_n \to A_n$ has one hump at k, and no others if and only if $f : (A_n \sim \{k\}) \to A_{n-1}$ has one hump at either $k - 1$ or $k + 1$. This means:

   $$N(n, k) = N(n-1, k-1) + N(n-1, k) = \binom{n-2}{k-2} + \binom{n-2}{k-1} = \binom{n-1}{k-1},$$

   by our induction hypothesis and a well-known property of binomial coefficients.

   Thus, the total number of bijections $f : A_n \to A_n$ with exactly one hump is $\sum_{k=1}^{n} N(n, k) = \sum_{k=1}^{n} \binom{n-1}{k-1} = 2^{n-1}$. Since the total number of bijections $f : A_n \to A_n$ is $n!$, then $P_n = \frac{2^{n-1}}{n!}$. Letting r be the unique integer such that $n = 2^r + q$ where $0 \leq q < 2^r$, then the exponent of 2 in $n!$ is equal to:

   $$\sum_{j=1}^{\infty} \lfloor \frac{n}{2^j} \rfloor \leq \sum_{j=1}^{r} \frac{n}{2^j} = n - \frac{n}{2^r} \leq n - 1,$$

   with equality in both places if and only if $n = 2^r$. Therefore, P_n is expressible as a unit fraction if and only if n is a power of 2, so the $n \leq 2020$ that satisfy this condition are precisely: $1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024$, of which there are 11.

2. Let Γ_1 and Γ_2 be externally tangent circles with radii $\frac{1}{2}$ and $\frac{1}{8}$, respectively. The line ℓ is a common external tangent to Γ_1 and Γ_2. For $n \geq 3$, we define Γ_n as the smallest circle tangent to $\Gamma_{n-1}, \Gamma_{n-2}$, and ℓ. The radius of Γ_{10} can be expressed as $\frac{a}{b}$ where a, b are relatively prime positive integers. Find $a + b$.

   Proposed by Adam Huang

   Answer: $\boxed{15843}$

   Note that the radii r_{n-2}, r_{n-1}, r_n satisfy the recurrence $\frac{1}{\sqrt{r_{n-2}}} + \frac{1}{\sqrt{r_{n-1}}} = \frac{1}{\sqrt{r_n}}$. Let $a_n := \frac{1}{\sqrt{r_n}}$. Then a_n obeys the Fibonacci recurrence with initial conditions $a_1 = \sqrt{2}$ and $a_2 = 2\sqrt{2}$. It follows that $a_{10} = 89\sqrt{2}$, so that $\frac{1}{\sqrt{r_{10}}} = 89\sqrt{2}$, $r_{10} = \frac{1}{89^2 \cdot 2}$ and our answer is $a+b = 1+89^2 \cdot 2 = 15843$.

3. A quadratic polynomial $f(x)$ is called *sparse* if its degree is exactly 2, if it has integer coefficients, and if there exists a nonzero polynomial $g(x)$ with integer coefficients such that $f(x)g(x)$ has degree at most 3 and $f(x)g(x)$ has at most two nonzero coefficients. Find the number of sparse quadratics whose coefficients lie between 0 and 10, inclusive.

   Proposed by Sunay Joshi

Answer: $\boxed{228}$

Let $N = 10$.

If $f(x)g(x)$ has exactly one nonzero coefficient, then $f(x)g(x) = cx^d$. Thus $f(x) = ax^2$ for $1 \le a \le N$, yielding $N = 10$ quadratics.

If $f(x)g(x)$ has exactly two nonzero coefficients, we proceed by casework on the degree of $f(x)g(x)$. Each case considers the smallest possible degree of $f(x)g(x)$ to ensure that the cases are distinct.

If the degree of $f(x)g(x)$ is 2, then $f(x) = ax^2 + bx$ or $f(x) = ax^2 + c$ for $a, b, c \neq 0$. This yields $2N^2 = 200$ quadratics.

If the degree of $f(x)g(x)$ is 3, we claim that $f(x) = km^2x^2 + kmnx + kn^2$ for $k, m, n \in \mathbb{Z}$, $\gcd(m, n) = 1$. To see this, suppose $f(x)g(x) = ax^3 + b = (cx^2 + dx + e)(\ell x + p)$. Dividing through by $a = c\ell$, it suffices to consider $x^3 + b = (x^2 + dx + e)(x + p)$ over the rationals. Expanding, we find $p + d = 0$, $dp + e = 0$, and $b = ep$. This implies $p = -d$, $e = d^2$, and $b = -d^3$. Thus $ax^3 + b$ must be of the form $km^3x^3 + kn^3$ with $\gcd(m, n) = 1$, and the claim follows.

In this case, note that $km^2 \le N$ and $kn^2 \le N$ implies $kmn \le N$. We proceed by casework on k.

If $k = 1$, then $m^2, n^2 \le 10$, hence $m, n \in \{1, 2, 3\}$. All pairs excluding $m = n \in \{2, 3\}$ are coprime, so we find $3^2 - 2 = 7$ solutions. If $k = 2$, then $m^2, n^2 \le 5$, hence $m, n \in \{1, 2\}$. All pairs excluding $m = n = 2$ are coprime, so we find $2^2 - 1 = 3$ solutions. If $3 \le k \le 10$, then $m^2, n^2 \le 1$, hence $m = n = 1$. This yields 1 solution for each value of k, hence 8 in total. Thus this case yields $7 + 3 + 8 = 18$ solutions.

Adding up, we find a total of $10 + 200 + 18 = 228$ solutions.

4. Find the largest integer $x < 1000$ such that $\binom{1515}{x}$ and $\binom{1975}{x}$ are both odd.

Proposed by Michael Gintz

Answer: $\boxed{419}$

Solution 1: Kummer's Theorem (special case): $\binom{n}{m}$ is odd iff you never carry while performing the addition of m and $n - m$ in base 2 (proof will not be provided, look online).

So, we need there to be no carries when we perform $x + (1515 - x)$ and $x + (1975 - x)$ in binary.
$1515 = 1024 + 512 - 21 = 10111101011_2$ and $1975 = 2047 - 64 - 8 = 11110110111_2$.
For there to be no carries, we need x and $1515 - x$ to both have 0's in the spots where 1515 has a 0. To maximize x, we should include as many 1's as possible. So, we should take x be as large as possible such that it has a 1 at each position where both 1515 and 1975 have 1's.
Then, $x = 00110100011_2 = \boxed{419}$.

Solution 2:

Lucas's Theorem: $\binom{n}{m} \equiv \prod_{i=0}^{k} \binom{n_i}{m_i} \pmod{p}$, with n_i is n's ith digit, base p. $\binom{n_i}{m_i} = 0$ if $m_i > n_i$.

Similar reasoning to before, if we want $\binom{n}{m} \equiv 0 \pmod{2}$, we need $\binom{n_i}{m_i} \equiv 1$ for all i, meaning $m_i = 0$ if $n_i = 0$. So, x can only have a 1 where both 1515 and 1975 have 1's, and the rest of the solution follows as before.

5. Let S denote the set of all positive integers whose prime factors are elements of $\{2, 3, 5, 7, 11\}$. (We include 1 in the set S.) If

$$\sum_{q \in S} \frac{\varphi(q)}{q^2}$$

can be written as a/b for relatively prime positive integers a and b, find $a+b$. (Here φ denotes Euler's totient function.)

Proposed by Sunay Joshi

Answer: $\boxed{1537}$

Since φ is multiplicative, the desired sum equals

$$\prod_{p\in\{2,3,5,7,11\}} \sum_{k\geq 0} \frac{\varphi(p^k)}{(p^k)^2}$$

We now consider the inner sum. Note that $\varphi(p^k) = p^{k-1}(p-1)$ for $k \geq 1$, while $\varphi(p^0) = 1$. Hence the sum reduces to

$$1 + \sum_{k\geq 1} \frac{p-1}{p^{k+1}} = 1 + \frac{p-1}{p^2} \cdot \frac{1}{1-1/p} = \frac{p+1}{p}$$

Therefore the desired sum equals

$$\frac{3}{2} \cdot \frac{4}{3} \cdot \frac{6}{5} \cdot \frac{8}{7} \cdot \frac{12}{11} = \frac{1152}{385},$$

and our answer is 1537.

6. Let $f(p)$ denote the number of ordered tuples $(x_1, x_2, \ldots, x_p)$ of nonnegative integers satisfying $\sum\limits_{i=1}^{p} x_i = 2022$, where $x_i \equiv i \pmod p$ for all $1 \leq i \leq p$. Find the remainder when $\sum_{p\in\mathcal{S}} f(p)$ is divided by 1000, where $\mathcal{S}$ denotes the set of all primes less than 2022.

   Proposed by Sunay Joshi

   Answer: $\boxed{475}$

   Considering the equation modulo p, we see that $\frac{p(p-1)}{2} \equiv 2022 \pmod p$, hence $p = 2$ or $p|2022 = 2 \cdot 3 \cdot 337$. It is easy to see that $p = 2$ yields zero solutions. Further, $p = 337$ yields zero solutions because the left hand side is at least $p(p-1)/2$. Hence the sum simplifies to $f(p)$ for $p = 3$. Subtracting i from x_i for $1 \leq i \leq p-1$ and dividing by p, we find the equivalent equation $\sum_{i=1}^{3} y_i = 673$ for $y_i \geq 0$. By stars and bars, this has $\binom{675}{2}$ solutions, which leaves a remainder of 475 when divided by 1000.

7. Alice, Bob, and Carol each independently roll a fair six-sided die and obtain the numbers a, b, c, respectively. They then compute the polynomial $f(x) = x^3 + px^2 + qx + r$ with roots a, b, c. If the expected value of the sum of the squares of the coefficients of $f(x)$ is $\frac{m}{n}$ for relatively prime positive integers m, n, find the remainder when $m+n$ is divided by 1000.

   Proposed by Sunay Joshi

   Answer: $\boxed{551}$

   The sum of the squares of the coefficients is $1+p^2+q^2+r^2$. By Vieta's formulas, $p = -(a+b+c)$, $q = ab+bc+ca$, and $r = -abc$. By independence, the expected value of the sum of the squares is therefore

   $$1 + (3\nu + 6\mu^2) + (3\nu^2 + 6\nu\mu^2) + \nu^3,$$

   where $\mu = \mathbb{E}(a)$ and $\nu = \mathbb{E}(a^2)$. It is easy to check that $\mu = 7/2$ and $\nu = 91/6$. Plugging these values into the above yields the fraction $1169335/216$, so that our answer is 551.

8. Let $\triangle ABC$ be a triangle with sidelengths $AB = 5$, $BC = 7$, and $CA = 6$. Let D, E, F be the feet of the altitudes from A, B, C, respectively. Let L, M, N be the midpoints of sides BC, CA, AB, respectively. If the area of the convex hexagon with vertices at D, E, F, L, M, N can be written as $\frac{x\sqrt{y}}{z}$ for positive integers x, y, z with $\gcd(x, z) = 1$ and y square-free, find $x + y + z$.

Proposed by Sunay Joshi

Answer: $\boxed{10043}$

Let $[\mathcal{P}]$ denote the area of polygon $\mathcal{P}$. Let a, b, c denote BC, CA, AB, respectively.

First note that the correct ordering of the vertices of the convex hexagon in counterclockwise order is $DNFEML$. Therefore we have the identity

$$[DNFEML] = [ABC] - [BND] - [CLM] - [AEF]$$

Since CLM is similar to ABC with $CL/CB = 1/2$, we have $[CLM] = \frac{1}{4}[ABC]$. Also, since AEF is similar to ABC with $AE/AB = \cos A$, we have $[AEF] = \cos^2 A \cdot [ABC]$. Finally,

$$[BND] = \frac{1}{2}[ABD] = \frac{1}{2}\frac{BD}{BC}[ABC] = \frac{c\cos B}{2a} \cdot [ABC]$$

By Heron's Formula, $[ABC] = \sqrt{9 \cdot 4 \cdot 3 \cdot 2} = 6\sqrt{6}$. By the Law of Cosines, $\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{1}{5}$ and $\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{19}{35}$. Plugging these values in, we find

$$[DNFEML] = 6\sqrt{6} \cdot \left[1 - \frac{19}{98} - \frac{1}{4} - \frac{1}{25}\right] = \frac{7587\sqrt{6}}{2450}$$

Our answer is therefore $7587 + 6 + 2450 = 10043$.

9. The real quartic $Px^4 + Ux^3 + Mx^2 + Ax + C$ has four different positive real roots. Find the square of the smallest real number z for which the expression $M^2 - 2UA + zPC$ is always positive, regardless of what the roots of the quartic are.

Proposed by Daniel Carter

Answer: $\boxed{16}$

Denote by $\Sigma_{a,b,c,d}$ the sum of the products of one root raised to the a, a different root raised to the b, a third root raised to the c, and the last root raised to the d. For example, if the four roots are p, q, r, s, then $\Sigma_{2,0,0,0} = p^2 + q^2 + r^2 + s^2$ and $\Sigma_{1,1,1,1} = pqrs$. We have that $U = -P\Sigma_{1,0,0,0}$, $M = P\Sigma_{1,1,0,0}$, $A = -P\Sigma_{1,1,1,0}$, and $C = P\Sigma_{1,1,1,1}$. Then one can see $M^2 = P^2(\Sigma_{2,2,0,0} + 2\Sigma_{2,1,1,0} + 6\Sigma_{1,1,1,1})$ and $UA = P^2(\Sigma_{2,1,1,0} + 4\Sigma_{1,1,1,1})$, so the expression $M^2 - 2UA + zPC$ is equal to $P^2(\Sigma_{2,2,0,0} + (z-2)\Sigma_{1,1,1,1})$.

Taking p, q, r, s arbitrarily close to each other makes $\Sigma_{2,2,0,0}$ close to $6p^4$ and $\Sigma_{1,1,1,1}$ close to p^4, so this expression is arbitrarily close to $P^2(z+4)p^4$. Thus if $z < -4$ this can be negative. Also, by AM-GM and the fact that the roots are all different we have $\Sigma_{2,2,0,0}/6 > \Sigma_{1,1,1,1}$, so if $z \geq -4$, the expression is positive. Thus $z = -4$ and our answer is $(-4)^2 = 16$.

10. The sum $\displaystyle\sum_{k=1}^{2020} k\cos\left(\frac{4k\pi}{4041}\right)$ can be written in the form

$$\frac{a\cos(\frac{p\pi}{q}) - b}{c\sin^2(\frac{p\pi}{q})},$$

where a, b, c are relatively prime positive integers and p, q are relatively prime positive integers where $p < q$. Determine $a + b + c + p + q$.

Proposed by Frank Lu

Answer: $\boxed{4049}$

We convert this into complex numbers, writing this as the real part of the sum $\sum_{j=1}^{2020} je^{\frac{4ij\pi}{4041}}$. Using the formula for the sum of a geometric series, we instead write this as $\sum_{j=1}^{2020}\sum_{k=j}^{2020} e^{\frac{4ik\pi}{4041}} = \sum_{j=1}^{2020} \frac{e^{\frac{8084i\pi}{4041}} - e^{\frac{4ji\pi}{4041}}}{e^{\frac{4i\pi}{4041}} - 1} = \frac{2020e^{\frac{8084i\pi}{4041}}}{e^{\frac{4i\pi}{4041}} - 1} - \frac{e^{\frac{8084i\pi}{4041}} - e^{\frac{4i\pi}{4041}}}{(e^{\frac{4i\pi}{4041}} - 1)^2}$. Now, we can rewrite this as $\frac{1010e^{8082i\pi/4041}}{i((e^{\frac{2i\pi}{4041}} - e^{\frac{-2i\pi}{4041}})/2i)} + \frac{e^{\frac{8080i\pi}{4041}} - 1}{((e^{\frac{2i\pi}{4041}} - e^{\frac{-2i\pi}{4041}})/2i)^2}$. Taking the real part of this yields the expression $\frac{1010\sin(\frac{8082\pi}{4041})}{\sin(\frac{2\pi}{4041})} + \frac{\cos(\frac{8080\pi}{4041}) - 1}{4\sin^2(\frac{2\pi}{4041})}$. But we know that $\sin(2\pi) = 0$, which means that we can rewrite this as $\frac{\cos(\frac{2\pi}{4041}) - 1}{4\sin^2(\frac{2\pi}{4041})}$. We thus get the answer $1 + 1 + 4 + 2 + 4041 = 4049$.

11. Let $f(z) = \frac{az+b}{cz+d}$ for $a, b, c, d \in \mathbb{C}$. Suppose that $f(1) = i$, $f(2) = i^2$, and $f(3) = i^3$. If the real part of $f(4)$ can be written as $\frac{m}{n}$ for relatively prime positive integers m, n, find $m^2 + n^2$.

Proposed by Sunay Joshi and Aleksa Milojevic

Answer: $\boxed{34}$

Note that Möbius transformations (such as f) preserve the cross ratio

$$(z, z_1; z_2, z_3) = \frac{z - z_2}{z - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2}$$

In particular, if $w = f(z)$, we must have $(z, 1; 2, 3) = (w, i; i^2, i^3)$. In other words,

$$\frac{z-2}{z-3} \cdot \frac{1-3}{1-2} = \frac{w - i^2}{w - i^3} \cdot \frac{i - i^3}{i - i^2}$$

Plugging in $z = 4$ and solving for w, we find

$$w = \frac{3}{5} - \frac{4}{5}i,$$

and so our answer is $3^2 + 5^2 = 34$.

12. What is the sum of all possible $\binom{i}{j}$ subject to the restrictions that $i \geq 10, j \geq 0$, and $i + j \leq 20$? Count different i, j that yield the same value separately - for example, count both $\binom{10}{1}$ and $\binom{10}{9}$.

Proposed by Nathan Bergman

Answer: $\boxed{27633}$

We refer to Pascal's triangle. Here, we'll solve the general case and plug in $10, 20$ at the end. It's tempting to look at this as rows with decreasing numbers of elements, but it is better to look at columns. To do so, change the problem to find the sum of all possible values of $\binom{i}{j}$ subject to the restrictions that $i, j \geq 0$ and $i + j \leq n$. This new problem has a sum of $2^n - 1$ (sum of Pascal's triangle rows below row n) larger than the previous problem, so we must subtract that at the end. Arranging this carefully, by looking at a sum of columns, we get:

$$\sum_{i=0}^{n} \sum_{j=i}^{2n-i} \binom{j}{i}$$

Using the hockey stick identity, we can turn this into

$$\sum_{i=0}^{n} \binom{2n+1-i}{i+1}$$

Letting $k = i + 1$, we can rewrite this as

$$\sum_{k=1}^{n+1} \binom{2n+2-k}{k}$$

By the Fibonacci identity for Pascal's triangle, we know

$$\sum_{k=0}^{n+1} \binom{2n+2-k}{k} = F_{2n+3}$$

Here, $F_0 = 0, F_1 = 1, F_m = F_{m-1} + F_{m-2}$ for $m \geq 2$. Applying the Fibonacci Identity to Pascal's Triangle gives our sum is $F_{2n+3} - 1$. Subtracting our initial $2^n - 1$ gives a final answer of $F_{2n+3} - 2^n$.

Lastly, the problem asks for $n = 10$, so the answer is $F_{23} - 2^{10} = 28657 - 1024 = 27633$.

13. Let $\triangle TBD$ be a triangle with $TB = 6$, $BD = 8$, and $DT = 7$. Let I be the incenter of $\triangle TBD$, and let TI intersect the circumcircle of $\triangle TBD$ at $M \neq T$. Let lines TB and MD intersect at Y, and let lines TD and MB intersect at X. Let the circumcircles of $\triangle YBM$ and $\triangle XDM$ intersect at $Z \neq M$. If the area of $\triangle YBZ$ is x and the area of $\triangle XDZ$ is y, then the ratio $\frac{x}{y}$ can be expressed as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

Proposed by Sunay Joshi

Answer: $\boxed{97}$

Below, let us relabel points T, D as points A, C, respectively. Let $a = BC$, $b = CA$, and $c = AB$.

Since $\angle YZB = \angle YMB = \angle XMC = \angle MZC$ and $\angle BYZ = \angle XMZ = \angle XCZ$, the triangles $\triangle ZYB$ and $\triangle ZCX$ are similar. The desired ratio is therefore $(YB/XC)^2$.

Since triangles $\triangle YBM$ and $\triangle YCA$ are similar, we have $YB/YC = MB/AC$ and $YM/YA = MB/AC$. Using the fact that $YC = YM + MC$ and $YA = YB + AB$, we find that $YB = \frac{MB^2(b+c)}{b^2 - MB^2}$. By symmetry, $XC = \frac{MC^2(b+c)}{c^2 - MC^2}$. Since M is the midpoint of arc BC, we have $MB = MC$, and hence the desired ratio reduces to $(\frac{b^2 - MB^2}{c^2 - MB^2})^2$.

To compute MB, note that $MB = \frac{a}{2\cos(A/2)}$. By the Law of Cosines,

$$MB^2 = \frac{a^2}{4\cos^2(A/2)} = \frac{a^2}{2(1+\cos A)} = \frac{a^2 bc}{(b+c)^2 - a^2}$$

Therefore the desired ratio equals

$$\left(\frac{b^2(b+c) - a^2 b}{c^2(b+c) - a^2 c}\right)^2 = \frac{81}{16},$$

and our answer is $81 + 16 = 97$.

14. Kelvin the frog is hopping on the coordinate plane $\mathbb{R}^2$. He starts at the origin, and every second, he hops one unit to the right, left, up, or down, such that he always remains in the first quadrant $\{(x, y) : x \geq 0, y \geq 0\}$. In how many ways can Kelvin make his first 14 jumps such that his 14th jump lands at the origin?

    Proposed by Ben Zenker

    Answer: $\boxed{613470}$

    Let $2L = 14$ be the length of the walk. Let $2k$ denote the number of jumps made to the left/right, so that $2(L-k)$ jumps are made up/down. The number of paths is therefore

$$\sum_{k=0}^{L} \binom{2L}{2k} C_k C_{L-k} \tag{1}$$

    where $C_k = \frac{1}{k+1}\binom{2k}{k}$ denotes the k-th Catalan number. We claim that the above is precisely $C_L C_{L+1}$, which for $L = 7$ equals $C_7 \cdot C_8 = 429 \cdot 1430 = 613470$, our answer.

    We now prove the claim. For convenience replace L with the variable n. Note that

$$\begin{aligned}
\binom{2n}{2k} C_k C_{n-k} &= \binom{2n}{2k} \cdot \frac{1}{k+1}\binom{2k}{k} \cdot \frac{1}{n-k+1}\binom{2(n-k)}{n-k} && (2)\\
&= \frac{(2n)!}{(k+1)!(n-k)! \cdot (n-k+1)!k!} && (3)\\
&= \frac{1}{(n+1)^2}\binom{2n}{n}\binom{n+1}{k}\binom{n+1}{k+1} && (4)\\
&= C_n \cdot \frac{1}{n+1}\binom{n+1}{k}\binom{n+1}{n-k} && (5)
\end{aligned}$$

    Summing over k and applying Vandermonde's identity, this becomes

$$\begin{aligned}
C_n \cdot \frac{1}{n+1}\binom{2n+2}{n} &= C_n \cdot \frac{1}{n+2}\binom{2(n+1)}{n+1} && (6)\\
&= C_n \cdot C_{n+1} && (7)
\end{aligned}$$

    as claimed. The result follows.

15. Let a_n denote the number of ternary strings of length n so that there does not exist a $k < n$ such that the first k digits of the string equals the last k digits. What is the largest integer m such that $3^m | a_{2023}$?

    Austen Mazenko

    Answer: $\boxed{9}$

    We claim that a_n satisfies the following recursive relations: $a_{2n+1} = 3a_{2n}$ and $a_{2n} = 3a_{2n-1} - a_n$. Such strings satisfying this criterion are known as *bifix-free*.

    We begin with the observation that if some string s is not-bifix free, then it's possible to find a $k \leq \frac{n}{2}$ such that the first k digits of s equals its last k digits. Suppose the length of the minimal substring s' of s that is both a prefix *and* suffix for it is length $k > \frac{n}{2}$. Thus, the prefix s' and suffix s' must overlap in $2k - n \geq 1$ values, so the last $2k - n$ digits of s' equal its first $2k - n$ digits. But, because s' is a prefix and suffix of s, we see that this means the first $2k - n$ digits of s equal its last $2k - n$ digits. We have thus found a substring s'' of length $2k - n < k$ that is both a prefix and suffix of s, contradicting minimality of s'.

    To see why $a_{2n+1} = 3a_{2n}$, notice first that if s is a length $2n$ bifix-free string, then inserting any digit right in the middle gives another bifix-free string s', because from our earlier observation if s' has any bifix, then it must have a bifix of length $\leq n$ which means it doesn't include the interpolated digit and thus would have been a bifix for s. Analogous reasoning show that any bifix-free string t of length $2n+1$ can be mapped to a bifix-free string t' of length $2n$ by removing its middle digit. This establishes a one-to-three mapping between length $2n$ and length $2n+1$ bifix-free strings, so $a_{2n+1} = 3a_{2n}$.

To see why $a_{2n} = 3a_{2n-1} - a_n$, we will demonstrate a one-to-three mapping between length $2n-1$ bifix-free strings and the union of the set of $2n$ bifix-free strings with the set of length $2n$ strings which are the concatenation of two copies of the same length n bifix-free string. First, for any bifix-free string s of length $2n-1$, we can insert any digit into its nth position 3 different ways. Now, the resulting length $2n$ string s' can't have any bifix of length $\leq n-1$ because then it would be a bifix of s. Thus, either s' is a length $2n$ bifix or it has a bifix of length n, aka, its first n digit substring equals its latter n digit substring. Moreover, we see this substring s'' must itself be bifix-free of length n because any bifix it has is a bifix of length $\leq n-1$ of s', but we showed this was impossible. It remains to see that any length $2n$ bifix-free string and any concatenation of a length n bifix-free string with itself can be constructed this way. Indeed, removing the nth digit from a length $2n$ bifix-free string must result in a bifix-free string, because if the result isn't bifix-free then it would have a bifix of length at most $n-1$ which would thus be a bifix of the original string. The same argument applies to the other case, whence the mapping is one-to-three, as claimed. Therefore, $3 \cdot a_{2n-1} = a_n + a_{2n}$.

Note $a_1 = 3, a_2 = 6$. To finish the problem, we remark that $\nu_3(a_n)$ is the number of ones in the binary representation of n. This can be proven by strong induction. The base cases obviously hold. Now, suppose it holds up to a_n. If n is even, then $n+1$ has one more binary 1 than n, and indeed $a_{n+1} = 3a_n \implies \nu_3(a_{n+1}) = 1 + \nu_3(a_n)$. If n is odd, then from the recursive relation for a_n we have $\nu_3(a_{n+1}) = \nu_3(3a_n - a_{(n+1)/2})$. If $\nu_3(3a_n) \neq \nu_3(a_{(n+1)/2})$, then we see $\nu_3(a_{n+1}) = \min\{1 + \nu_3(a_n), \nu_3(a_{(n+1)/2})\}$. Note that, to get from $(n+1)/2$ to n in binary, you append a 0 to the right, then replace all the trailing zeros with ones and the rightmost one with a zero. In particular, this process either keeps the numbers of ones the same or raises it. Thus, by the inductive hypothesis, $\nu_3(3a_n) \neq \nu_3(a_{(n+1)/2})$ always holds, so $\nu_3(a_{n+1}) = \nu_3(a_{(n+1)/2})$, which by the inductive hypothesis is precisely the number of ones in the binary representation of $(n+1)/2$ which equals the number of ones in the binary representation of $n+1$, as desired.

Thus, $\nu_3(a_{2023})$ is the number of ones in the binary representation $2023 = 11111100111_2$, namely, 9.